Promoting adults' knowledge for engaging young children with numerical activities

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Considering that young children spend a great deal of time outside of school, there is a need to increase the expertise of adults when it comes to engaging young children with numerical activities. This study presents the Cognitive Affective Mathematics Adult Education framework used to design such a course for interested adults. Focusing on promoting young children's competency for comparing sets, we illustrate how the framework was used to design the course and how the framework was used to analyze the impact of the course on the relevant expertise of the participating adults. The findings indicated that the adults had gained knowledge in the various strategies one can use to compare sets, had become aware of the importance of evaluating those strategies, and had acquired the ability to take the children's ages and abilities into account when tailoring activities that engage them with set comparisons.

Key words:
Adults' numerical knowledge, set comparison, young children.

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1 Introduction

This paper describes an intervention conducted with adults who are not preschool teachers, aimed at promoting their knowledge for engaging young children with numerical activities. While several studies have focused on early childhood teachers (e.g., Ginsburg, 2016; Tsamir et al., 2015), young children spend much of their time outside of school, with parents, grandparents, and other responsible adults. Thus, the home environment, along with the responsible adults who care for preschool children, can also have an impact on children's mathematical knowledge. Previous studies (e.g., Missall et al., 2015) have found that most adults believe in the importance of promoting early numerical competencies, but are not always aware of what and how to promote such knowledge (Cannon & Ginsburg, 2008). Thus, providing workshops, courses, and interventions for those adults is important.

In this paper we present part a large research project that aimed to investigate adults' knowledge and beliefs regarding playful mathematics learning during the early years (e.g., Levenson et al., 2022). As part of the project, we planned and implemented an intervention with adults that focused on enhancing adults' knowledge of playful numerical activities. The aim of this study is to explore in what ways an intervention for adults can impact on adults' knowledge for engaging young children with numerical activities.

2 Literature review

2.1 Promoting preschool children's numerical competencies

Early childhood researchers acknowledge that young children naturally engage in several numerical activities, such as counting objects, comparing sets, and number composition (Baroody, 1984). Object counting, also called enumeration, refers to counting objects for the purpose of saying how many and includes the stable-order principle, the one-to-one principle, the cardinality principle, the abstraction principle, and the order-irrelevance principle (Gelman & Gallistel, 1978). The stable order principle, in essence, refers to verbal counting, that is, reciting the conventional number words, producing them in order, and consistently. One-to-one correspondence involves assigning one count word to each object (Sarama & Clements, 2009). At the core of this skill is an understanding that each element in one set can be paired with one, and only one, element in a second set. One-to-one correspondence is not only used in counting. When comparing the number of items in two sets, children may match each item of one set to each item of another set. In one study (Tsamir et al., 2010), for example, children were asked to assess which of two piles of identical bottle caps had more. One pile had 12 bottle caps and the other had nine. One child took the two piles of caps and lined them up in two separate rows, making sure that each cap touched the next, and then compared the length of each row. Researchers also found that one-to-one correspondence between items in two sets comes more naturally to children when the items contextually belong together, such as baby animals to adult animals (Greenfield & Scott, 1986) or straws and cups (Clarke et al., 2006). The cardinal principle refers to knowing that the last number said when counting objects represents the quantity, the abstraction principle is knowing that any set of discrete objects can be counted, and the order-irrelevance principle means that objects can be counted in any order (Gelman & Gallistel, 1978).

Related to enumeration is the process of subitizing (Baroody, 2004). Subitizing refers to the immediate recognition of the number of items in a collection without counting the objects. It involves recognizing that a picture of three dots set up in a triangular pattern has the same number of dots as a row of three dots and that both sets have exactly three dots. As children develop, these conceptions may be mentally operated on, such as mentally decomposing a pattern of five into three and two and combining them back together (Sarama & Clements, 2009). As the number of items in a collection grows, perceptual subitizing may again be helpful.

Recognizing the importance of preschool children's engagement with numerical activities, several educators have suggested ways for enhancing these competencies. For example, playing hide-and-seek, can be a way of practicing verbal counting (Gunderson & Levine, 2011). Body movements can also be used to help children differentiate decades from 1 to 100, such as making funny faces when reciting numbers from 11 till 19 and twisting your body when reciting twenties (Greenes et al., 2004). To encourage different enumeration sub-competencies, researchers have suggested paying attention to the types of items being enumerated, as well as the way they are set up. For example, counting objects that are different in function, color, or size, can help children recognize the abstraction principle and that these other attributes do not affect counting (Greenes et al., 2004). Counting objects in different formations may promote the order-irrelevance principle (Gelman & Gallistel, 1978). In Tsamir et al.'s (2018) study, when 4–5-year-old children were requested to count seven bottle caps placed in a circle, two children who had previously correctly counted the bottle caps when placed in a row, claimed that they did not know what to do and gave up. Setting a table for a meal can be a way for even toddlers to practice one-to-one correspondence, setting one plate for each chair, and for each plate one cup (Tirosh et al., 2020).

Adults' number talk has also been shown to be important when aiming to enhance children's number knowledge. Children whose parents frequently used number words when counting and labeling a set size, had a greater understanding of cardinality than children whose parents hardly used number words (Levine et al., 2010). Labeling the set size emphasizes the cardinality principle. This significant point was stressed by Gunderson and Levine (2011) who further investigated the variation in parents' number talk, whether they labelled the cardinal values of sets, whether the objects counted were present, as well as the set size (up to three objects or more). They found that number talk referring to large sets of present objects contributed to children's cardinal-number knowledge, and that talking mostly about small numbers can delay the development of cardinality.

In Israel, where this study took place, there is a mandatory national mathematics preschool curriculum Israel National Mathematics Preschool Curriculum [INMPC], 2010. The curriculum lists competencies that children should meet before entering first grade, as well as suggested activities for promoting those competencies. Related to this study are counting, enumerating, and set comparison. According to the curriculum, children entering first grade should be able to count verbally till 30, enumerate a set of 20 objects, and compare sets of various amounts. Furthermore, comparing sets, according to the curriculum can be carried out by using one-to-one correspondence, estimation, and enumeration. As for suggested activities, the curriculum relates to the kindergarten setting. For example, when handing out musical instruments during music time, the teacher can ask, "Are there enough tambourines so that each child will have one? How can we know if there are more tambourines or more children?" These questions encourage one-to-one correspondence. The curriculum goes on to state that asking a different question, such as "How many tambourines do we need if each child is to receive one tambourine?" will encourage set comparison by enumeration. Finally, the curriculum notes that we can first ask the child to estimate and then check the estimation by one-to-one correspondence or enumeration. In other words, children should be encouraged to compare sets using various methods.

2.2 Adults' knowledge regarding children's numerical competencies

The current study focuses on adults who are not preschool teachers. Previously, studies related to adults' knowledge of young children's numerical competencies offered adults a list of activities and inquired the frequency of their occurrence in the home. For example, Missall et al. (2015) and Skwarchuk (2009) found that the most frequent activities reported by parents were counting aloud, counting out a number of items from a larger group, and reading numbers. Among the least frequent activities were counting backwards, skip counting, and comparing the number of objects in two sets. Another study found that parents engaged their children with identifying numerals more often than comparing the magnitudes of numbers (Vandermaas-Peeler et al., 2012). In a recent study (Barkai et al., 2022), adults were given open questions and asked which numerical competencies can be promoted in early childhood. Like previous studies, the most frequent competency mentioned was counting, although it was not always clear if participants meant verbal counting, object counting, or both. When asked to describe situations where they observed children engaging in numerical activities, again, the most prevalent competency mentioned was counting.

From participants' descriptions it was evident that participants meant object counting. In a follow-up study (Levenson et al., 2021), adults were requested to suggest tasks that could promote various number competencies among young children. Many of the adults did not differentiate between verbal counting and object counting. Furthermore, when suggesting enumeration activities, less than half of the suggestions paid attention to enumeration principles, such as one-to-one correspondence or cardinality. The authors suggested that workshops for adults could point out why the details of an activity are important, and increase adults' repertoire of activities for furthering children's number skills.

A few studies observed parents in action. When parents were requested to record themselves engaging their young children with mathematics at home, the most prevalent activity recorded was object counting (Anderson & Anderson, 2018). Interestingly, in everyday situations where numerical ideas might be discussed, such as shopping, it was found that parents barely used the opportunity to encourage skills such as counting or addition (Vandermaas-Peeler et al., 2009). Another study (Vandermaas-Peeler et al., 2012) found that when cooking with their children, most parents did not engage their children with numerical skills until they were prompted to do so. However, with guidance and prompting, parents introduced addition and subtraction problems, as well as the notion of cardinality.

Recognizing that children from low-income families enter school with less mathematical knowledge than their peers (Reardon & Portilla, 2016), several studies initiated mathematical interventions for disadvantaged families. Starkey and Klein (2000) investigated the impact of one such intervention. Families received math kits to use at home, and a designated parent attended classes with their pre-kindergarten child at a local center. The kits contained sets of activities with concrete materials that could be used to support children's mathematical development. In the end, children whose parents attended the intervention developed more mathematical knowledge than children in a control group. In another study (Sonnenschein et al., 2016), parents were given instructions of how to use a familiar board game to promote counting competencies. Findings indicated that most parent did not play the game at home according to the suggested instructions. In an effort to increase the success of home interventions, some educators implemented programs that included both parents and preschool teachers (e.g., Marti et al., 2018). However, not all preschool teachers believe in the importance of family involvement (Tirosh et al., 2017). While parents are those who perhaps have the most impact on their children's mathematical growth outside of school, in some cultures, it is the grandparent or aunt, who cares mostly for the young child. Thus, in the current study, we consider an intervention designed for all interested adults, not only those who are parents.

3 Theoretical framework

In this study, we describe an intervention offered to adults who were not preschool teachers. The design of the intervention was based on our previous work with preschool teachers. With preschool teachers, we used the Cognitive Affective Mathematics Teacher Education (CAMTE) framework when investigating and promoting teachers' knowledge and self-efficacy for teaching number, geometry, and pattern concepts (e.g., Tsamir et al., 2014). Expanding on previous studies of teachers' knowledge (e.g., Shulman, 1986), the CAMTE framework differentiated between two aspects of teachers' subject-matter knowledge (SMK): being able to produce solutions, strategies and explanations and being able to evaluate given solutions, strategies, and explanations. The CAMTE framework also differentiated between two aspects of pedagogical content knowledge (PCK): knowledge of content and students (KCS) and knowledge of content and teaching (KCT) (Ball et al., 2008). In our current research, we consider adults who are not professional educators and the mathematics knowledge and beliefs we wish to foster among those adults who might then engage young children, out of school, in mathematical activities (see Tab. 1, Cells 1 and 2).

Because we are considering adults who are not teachers, instead of referring to PCK, we relate to knowledge needed for engaging children with playful mathematics (henceforth, Mathematics Engagement Knowledge) (Cells 3 and 4). Instead of KCS, we focus on knowledge of content and children, such as knowing that children aged three may not yet have acquired the cardinality principle of counting. Instead of KCT, we relate to knowledge of content and playful learning, meaning knowledge of activities that can promote numerical thinking in a playful manner. Our adapted framework for adults who are not preschool teachers, is called the Cognitive Affective Mathematics Adult Education (CAMAE) framework.

Each knowledge cell has a corresponding belief cell. Whereas for teachers we were interested in their self-efficacy for teaching mathematics, in our study with adults, we were interested in their beliefs regarding what mathematics children (and the adults who interact with them) should know, and how children can engage with mathematics. That is, we consider beliefs related to mathematics (Tab. 1, Cells 5 and 6) as well as engagement beliefs, i.e., beliefs related to engaging children with mathematics (Tab. 1, Cells 7 and 8).

Tab. 1: The Cognitive Affective Mathematics Adult Education (CAMAE) Framework

	Mathematics for adults and children		Mathematics Engagement	
	Solving	Evaluating	Children	Playful learning
Knowledge	Cell 1: solving tasks. e.g., count the number of elements in a set using a variety of strategies	Cell 2: evaluating tasks. e.g., evaluate the efficiency of a counting strategy	Cell 3: knowledge of children's conceptions. e.g., which number symbols do children confuse.	Cell 4: knowledge of content and playful learning. e.g., which activities can foster children's acceptance of the one-to-one principle.
Beliefs	Cell 5: mathematics beliefs related to solving tasks. e.g., is it important to know several ways to count the number of items in a set.	Cell 6: mathematics beliefs related to evaluating tasks. e.g., is it important to know which solution methods are efficient.	Cell 7: beliefs regarding children and mathematics. e.g., believing that young children enjoy learning number concepts.	Cell 8: beliefs regarding ways of engaging children with playful mathematics. e.g., believing that adult guidance can foster the learning of early number concepts.

In a previous study (Levenson et al., 2021), we reported on adults' beliefs related to Cells 5, 6, 7, and 8. For example, related to Cells 5 and 6 we asked participants if they believed it was important for children to be able to solve number tasks in various ways, and if they believed children should be able to choose appropriate ways for solving numerical tasks. Related to Cells 7 and 8, we asked adults if they believed children's number knowledge could be promoted and if interactions between a child and an adult while engaging in an activity/game can increase the child's knowledge of number. In general, beliefs were very positive. Furthermore, no differences were found between parents, adults who have some other connection with young children, and those who claimed to have no connection with young children. In the current study, we focus on adults' knowledge, Cells 1–4 of the framework.

4 Methodology

4.1 Sample

The current study took place at our university. As part of the graduate program in mathematics education, we offered students the opportunity to take an elective course entitled, "Numerical Thinking in Early Childhood". The course aimed to promote participants' knowledge and awareness of number competencies developed prior to first grade, as well as the tasks that might promote early number knowledge and competencies. Eighteen graduate students participated in the course, none of whom were preschool teachers. Seven participants were parents of young children, eight were aunts or uncles, and three claimed to have no immediate contact with young children. We chose this context for our study, wishing to include at this point in our research only adults who we hypothesized would have a general positive disposition towards mathematics.

4.2 Data collection

The course consisted of 13, 90-minute sessions and focused on three major numerical competencies: counting and enumerating, comparing sets, and number composition and decomposition. In this study, we focus on the second topic, comparing sets. In a previous study (Barkai et al., 2022), only a fifth of the participants mentioned set comparison as a competency that can be promoted among young children. This topic was introduced during the sixth lesson, after participants had discussed counting and enumeration, and had gained some experience analyzing children's knowledge of counting and enumerating, as well as designing counting and enumerating activities. In general, during the course, participants engaged in a series of activities that were repeated for each competency: (1) read and discuss related research (e.g., Gelman & Galistel, 1978; Tsamir et al., 2015), (2) view YouTube videos of preschool children practicing that competency and/or engage in class with related activities, (3) individually design a task to implement with a young child aimed at promoting that competency, (4) discuss together proposed tasks and agree upon one common task that each participant would implement, (5) implement the agreed upon task with a child while videoing the activity, (6) individually analyze the child's knowledge and competency when carrying out the task, (7) view and analyze together participants' videos.

As an explorative study, we take a qualitative approach to the research. Data was collected in a natural setting, and reviewed by the authors of this paper, who also developed the instruments. Some of the instruments used (e.g., see Fig. 1) had been used in previous studies and were shown to be valid. Each course lesson was led by the leading author of this paper, with the second author present as well. The lessons were all recorded and transcribed. Ethical approval was given by the university's institutional review board. In the following section, we present and describe two lessons dealing with the topic of set comparison.

4.3 Data analysis

Using the CAMAE framework as an analysis tool, we investigate how different knowledge elements (Cells 1–4) were advanced in each of the lessons. Thus, although it may be said that the data is interpretive in nature, following a framework for analysis, allows researchers a common lens with which to view the data. The first two authors analyzed the data separately and then compared analysis and interpretations. The third author then validated the final outcomes.

5 Results

5.1 Introducing set comparison – Lesson 6

When dealing with young children, comparing sets most often refers to sets that contain a finite number of physical objects. Thus, not surprisingly, in a previous study with preschool teachers (Tirosh et al., 2011), it was found that when asked to explain how one can know which set has the most elements, nearly all teachers reported that one must count the elements of each set, not mentioning other methods such as using one-to-one correspondence or estimation. Yet the preschool curriculum (INMPC, 2010) specifically stipulates that children should learn more than one method of comparing sets. As an introduction to the topic of set comparison, participants filled out a closed questionnaire (based on Tsamir, 1999) related to set comparisons (see Fig. 1).

1. Here are two sets A and B:
$$\mathsf{A} = \{1; t; a\}, \qquad \mathsf{B} = \{7; w\}$$
Is the number of elements in sets A and B equal? Yes/No Explain
2. At a dance party, all the students danced in couples, a boy and a girl in each couple. No pupils were left without a partner.
$$\mathsf{Z} = \{\text{the boys}\}, \qquad \mathsf{W} = \{\text{the girls}\}$$
Is the number of elements in set Z equal to the number of elements in set W? Yes/No Explain.
3. Given the sets:
$$\mathsf{X} = \{1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12\} \qquad \mathsf{Y} = \{a; b; c; d; e\}$$
Is the number of elements in set X equal to the number of elements in set Y? Yes/No Explain.
4. Given the sets:
$$\mathsf{P} = \{a; b; c; d; e\} \qquad \mathsf{V} = \{a; b; c\}$$
Is the number of elements in set P equal to the number of elements in set V? Yes/No Explain.
5. Dan was ill. The doctor prescribed one green tablet every 3 hours for the first week. Then, in the second week, he was ordered to take a red capsule every 3 hours.
$$\mathsf{G} = \{\text{The green tablets}\} \qquad \mathsf{R} = \{\text{The red capsules}\}$$
Is the number of elements in set G equal to the number of elements in set R? Yes/No Explain.

Fig. 1: Are the sets equivalent?

The aim of the questionnaire was to raise to the fore the notion that comparing sets need not always be carried out by counting. As can be seen, for some of the sets, the number of elements can be counted (see questions 1,3, 4, and 5), for some, the number of elements can be counted but it is not the most efficient way to compare the sets (see question 5), while for other sets, the elements cannot be counted (see question 2).

Participants' explanations were categorized according to the strategies they indicated: enumeration, one-to-one correspondence, estimation, and other. As can be seen from Tab. 2, some of the questions elicited a nearly unanimous strategy (e.g., one-to-one correspondence for question 2). On the other hand, participants seemed to be split on the strategy used for comparing sets in questions 3 and 4. Responses categorized as 'other' basically just wrote which set was bigger, without explaining how they knew it. For example, for the third question, a participant was categorized as 'other' because she wrote, "There are less elements in set Y."

Tab. 2: Strategies used when comparing sets ($N = 18$)

Question	Enumeration	One-to-one correspondence	Estimation	Other
Q1	14	2	2	–
Q2	2	16	–	–
Q3	8	1	8	1
Q4	8	3	7	–
Q5	–	13	3	2

After all participants filled in the questionnaire, each question was discussed in terms of its potential to elicit different numerical competencies. For example, focusing on the two sets in the first question that contain both letters and numbers, Esther commented, "This problem illustrates one of Gelman and Gallistel's counting principles – the abstraction principle – that all things can be counted." Surprised that so many participants wrote that they counted the elements in each set, Esther asked someone to explain why he or she used this strategy. Levy (all names other than Esther and Ruthi are pseudonyms) replied, "I wanted to be sure. There are open parenthesis and closed parenthesis, and commas and different letters, and so I wanted to be absolutely sure, so I counted, enumerated and that's it." Riva offered a different explanation, "I wrote that I counted but really I subitized, and that's like sort of enumeration because I know the amount, I know that here are two and here are three, so isn't subitizing in a way like enumerating?" As seen by the participants' comments, the set comparison questions led back to the topic of enumeration and the use of subitizing, reinforcing that new competencies, such as comparing sets, lean on previously acquired competencies.

Regarding questions 3 and 4, nearly half of the participants wrote that they estimated and nearly half wrote that they counted the objects. While discussing these options, the following arose:

> Leo: From just looking at it (sets X and Y in question 3), I estimated, because I see that the left set is bigger than the right set.
>
> Esther: How do you know that? How is it so clear to you that it's bigger?
>
> Leo: Because you see right away that the row on the left is longer and so it's clear that the left side has more objects.

At this point, Esther takes out some bottle caps and arranges them in two groups, first as seen in Fig. 2a, and then like in Fig. 2b.

a) 

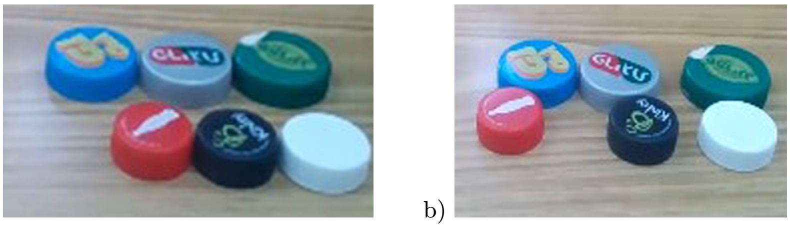
b) 

Fig. 2: Which set has more bottle caps?

> Esther: Do you still think that your strategy is valid?
>
> Mohammad: She (Esther) stretched the three caps to the length of six caps.

Of course, as an adult, Leo would know that the sets in Fig. 2b are not equivalent. Esther's demonstration was not to test Leo's number conservation ability, but to illustrate why the mere explanation of one row being longer than another row, is not acceptable. This is related to Cell 2 of the framework, knowing how to evaluate strategies. Esther then goes on to demonstrate another situation, where the caps in one row (one set) have a larger surface area than the caps in the second row (second set). Also in that case, Leo's explanation would not be acceptable.

Esther: And what about in this situation? (See Fig. 3a) Can we still know which set has more by comparing the length of the line? Do I need to count? (Esther moves the caps to the position shown in Fig. 3b.) Or can I use one-to-one correspondence?



a) b)

Fig. 3: Which set has more bottle caps?

The above demonstration also raises the importance of the objects being counted. In the first demonstration of bottle caps (see Fig. 2a and 2b) the caps are all the same size, while in the second set the caps are different sizes. This provides an opportunity for discussing critical attributes.

Esther: The fact that the bottle caps are different sizes brings to our attention attributes, such as color and size, that need to be ignored when counting objects.

To summarize, the sets presented on the questionnaire demonstrated, first of all, that there are several ways to compare sets, an application of Cell 1 of the CAMAE Framework. Discussing the ways participants solved the questions, and evaluating different strategies for comparing sets, was an application of Cell 2 of the framework. Finally, the set comparison activity allowed participants to not only review enumerating principles learned during previous sessions, such as one-to-one comparison, but to see how such principles could be applied to other numerical skills, such as comparing sets.

5.2 Designing and analyzing a set comparison activity – Lesson 8

After lesson six described above, participants were given the assignment of designing a playful activity for engaging young children with comparing sets. These activities were then presented and discussed during lesson eight, two weeks later. Below, we describe a discussion that centered around one of the participant's activities, and present the impact of that discussion on adults' knowledge of children's conceptions (Cell 3 of the CAMAE framework) and knowledge of content and playful learning (Cell 4 of the CAMAE framework).

Michelle presented the following activity during the eighth session:

I thought about doing a birthday party, placing the plates nicely on the table. And I thought of using a small number of plates. Around five, or something like that. And then I would put a bunch of items, for example, small prizes like a whistle, a lollipop, a small bag of goodies, all sorts of things like that. Not too big, so you can see with your eyes the amount. And then I would ask the child: Is there enough? Is there not enough? So he would really check if the sets went together, if the amount of plates is equal to the number of items, so that all the children who came to the party could get everything.

Michelle sets her activity in a playful and fun context for children, that of a birthday party. She does not offer many details, nor does she even mention what exactly the child will be comparing. To clarify the activity Esther comments:

Esther: So, we compare the amount of plates to the amount of prizes?

Michelle: Yes. And each time we use a different amount of prizes. I also think that it's a feast for your eyes. Something nice.

Michelle is quite excited about the context, but still does not offer details. Finally, after additional prompting, Michelle suggests that for one of the prizes, the amount could be greater than the number of plates, for a different prize the amount could be equal to the number of plates, and for a different prize, the amount could be smaller than the number of plates. The rest of the participants then begin to comment:

Dave: I think the activity is nice. There is one-to-one correspondence, three different examples, equal, greater than, and less than. It's not too difficult. You don't need complex materials, and the child will be attentive because there are whistles and all sorts of prizes and colors.

Izzie: With these amounts (referring to Michelle's initial statement that she would use five plates), you could just subitize. What age are we talking about? Five years? At that age they could subitize.

Esther: No one said how the items are placed.

Izzie: Four whistles can be subitized regardless of how they are placed.

At this point, the participants have begun to think of the children. Dave considers a young child's attention span and interest in the activity, as well as how a child might solve the problem of where there is more (Cell 3 of the CAMAE framework). This is related to how the birthday table is set up (Cell 4 of the framework) and knowing that the details of an activity can foster different aspects of a child's numerical ability, such as subitizing. This is further elaborated by Tom, who adds another way of comparing sets: "You can disperse part of the items, and then we gain a few things, space and visibility, and also estimation."

In addition to the way the items are set up, the wording of questions we ask children, or the instructions given, can elicit different competencies. This is taken up by Mike.

Mike: I don't see here the need for counting or enumerating because you can just put one candy on each plate and when there are no more, the child will say there aren't enough, and if he has enough, he will say that there is enough. So, he doesn't even know how many plates and candies there are.

Nomi: But that's not the goal here. On the contrary. The goal is not to have the children count or enumerate.

Mike: But where is the set comparison here? He doesn't know in the end how much he has because he didn't need to count.

Ruthi: If I have leftover, if I use one-to-one correspondence, isn't that comparison?

Mike: In the beginning, you just asked, "Are there enough candies so that you can put one on each plate?" And so, he puts one candy on each plate. You should ask, "Are there more plates or candies?" And then there are many ways to compare.

Mike's comment is directed at Michelle's original activity and the question she said she would pose to children – Are there enough candies to put one on each plate? On the one hand, the question is open, and does not specifically direct children to do a particular act. On the other hand, in the wording "Are there enough candies to *put one on each plate*," there is a hint to use one-to-one correspondence. Furthermore, it seems that Mike still considers enumerating as a necessary part of set comparison, which leads both Nomi and Ruthi to remind everyone that the aim is not to know how many, but to know which set is bigger, plates or candies. As Mike relents and agrees that there are many ways to compare sets, he offers a different question ("Are there more plates or candies?"), which in his opinion, would be more open.

To summarize, Cells 3 and 4 of the framework are about knowing how a child might engage with a numerical activity and how an activity might elicit certain competencies. The discussion above revolved around both issues. Participants thought about how children might engage in the task, whether they would use subitizing and/or one-to-one correspondence, and how the setup of the activity might elicit different responses. Furthermore, participants realized how the questions we ask, or the instructions given to a child, might also elicit different competencies.

5.3 Adopting, adapting, and extending the birthday party activity

After discussing Michelle's activity, participants were requested to try out the activity with a young child, video the activity, and analyze the child's knowledge and strategies used to compare the sets. They were then asked to reflect on the activity and what they were still curious about. Below, we offer excerpts from several participants' reports and further investigate elements of adults' knowledge that came to the fore.

All participants, except for two, adopted the birthday party theme, using plates (colored hard plastic plates or paper plates) and some other party related object (candies, biscuits, candles, or chocolates). Regarding the types of objects used in the activity, one participant, Seena, commented, "The child was constantly busy with the candies and wanted to finish quickly so he could eat the candy. Next time, I wouldn't use candies because he wasn't focused on the activity." In other words, while participants had agreed that the birthday party theme was playful and that it would raise children's motivation to participate in the activity, they were also willing to admit when a change of objects was necessary.

Regarding how objects were laid out, we noted a variety of setups: each group of items was spread out on different sides of the table (see Fig. 4), each group was bunched together in separate piles, or one group was bunched up and one was laid out in a row. Michelle, who used cup and straws, explained how she chose her setup, "I thought about it in advance, to place the cups in a group and the straws in a row. Because my son is comparatively old (he's five), I wanted to raise the level of difficulty and thought about not placing the items one directly in front of the other." Not wanting to make one-to-one correspondence immediately obvious, Michelle considers the layout of the objects. Another participant (Natasha), who had spread out cups and coffee capsules on the table, reported that her child placed one coffee capsule in each cup when deciding which set had more. She was curious and wrote, "I wonder what the child would have done if the cups were placed one inside another." Natasha is also thinking about how the setup might impact on the child's actions" (Cells 3 and 4 of the framework).

Fig. 4: Are there more plates or candles or is there the same amount?

Recall that participants had discussed what exactly to ask the children during the activity. From their written reports it became evident that three variations of the task question were asked: What is there more of? What is there more of, plates or candies (or candles or chocolates)? What is there more of, candies or plates or is there the same amount? Dave, who reported using the third variation of the question commented, "[I wonder] what would happen if I just asked where is there more, or if I asked, is there one group which is bigger?" Dave was referring to the situation where both groups (in his case plates and candies) had the same number of items. He further reflected, "The question is a bit confusing, because it seems like there must be one group which is bigger than the other." Another participant, Aamil, also used the third version of the question. His young niece (three-years old) seemed to just guess, often with incorrect responses. He reflected, "I am interested to know what would happen if *after* I asked her which set had more items, I would ask her how many plates there are and how many candles there are. Maybe then she would count." Both Dave and Aamil show recognition of the importance of questioning when engaging children with numerical activities.

Finally, we note that participants took notice and commented on specific strategies used by children to compare the sets, as well strategies used when enumerating the objects. For example, Izzie wrote, "The girl put each chocolate on a plate and counted as she went along. She understood cardinality, connecting numbers to amounts." Izzie noted both the principle of one-to-one correspondence, as well as the principle of cardinality. Tamar wrote: "During the game, (after the situation where the amounts were equal) I added more plates in front of her eyes, and she still counted each set to find out where there more." Tamar's description, and her used of the term 'still', hints at her amazement at the girl's reaction to a seemingly simple task. Mary noted that the boy she played with knows how to enumerate and compare sets, but "he doesn't know how to estimate at first glance. He asks each time if he can count first." Interestingly, both Tamar and Mary are commenting what the children they played with did not do, in addition to what they did. They are evaluating the children's strategies (Cell 2 of the framework).

Dave was curious about the physical actions taken by the child when enumerating. He wrote, "The child first put all of the candies in his hand, and then counted them as he put each one on the floor (the activity took place while sitting on the floor). After he finished, he started to count the plates; he counted out loud, 1, 2, 3 as he picked up each plate and then he looked at the two left, went to pick up the fourth, stopped and thought (or calculated) a minute and announced, "it's the same." Dave then commented on the boy's actions: "Besides the comparison of sets, it's interesting to note his enumeration technique. With the candies he takes each one out of his hand and with the plates, he picks up each one and holds it in the air. There is also the question of how did he know that there were five plates? Did he add the two to the three or did he count in his head?" The detailed account written by Dave indicates his keen observation, which leads him to ponder what the boy was thinking.

To summarize, each participant engaged a young child with the comparison activity, which allowed them to try out a playful activity and see for themselves how the various details of an activity can impact on what a child does. This is related to Cells 3 and 4 of the CAMAE framework. Their reports on the activity, as well as their evaluations of the children's strategies, also revealed knowledge regarding enumeration principles as well as comparison strategies (Cells 1 and 2 of the framework).

6 Discussion

The aim of this study was to explore the ways an intervention for adults can impact on adults' knowledge for engaging young children with numerical activities. Using the CAMAE framework, we designed a course for adults interested in supporting early childhood mathematics. To engage children with playful activities that can encourage mathematical growth, adults first need to know what numerical competencies are appropriate for children at different ages, and what prior knowledge is necessary for each competency. Previous studies showed that adults are not aware of the various numerical competencies that can be promoted during early childhood, such as set comparison (Barkai et al., 2022). Thus, a first aim of interventions with adults is to raise their awareness of the various mathematical skills that can be nurtured before first grade.

Yet, raising awareness may not be enough if adults do not know what is involved in those competencies. A challenge working with adults on early mathematical competencies is helping them recognize the complexity of those competencies for children and the sub-competencies necessary for carrying out a numerical procedure. By first presenting them with a set-comparison questionnaire, seemingly not related to young children, we were able to raise their mathematical knowledge in a respectful way.

An important part of the intervention was having participants design activities themselves, instead of just giving them the activities to implement with children, as was done in other interventions (Starkey & Klein, 2000). Designing the activities provoked participants to carefully consider the physical items to be used in an activity, and also allowed them to see how everyday items that are found in nearly every home, can be used to foster numerical competencies. Indeed, one of the participants complimented another's activity because it did not use complex materials. Furthermore, discussing the designed activities led to further inquiry regarding the specific instructions to be given to the child and the specific questions the adult should ask the child during activity. This was an important element as previous studies have shown that merely offering parents activities does not guarantee they will follow the intended instructions (Sonnenschein et al., 2016). Furthermore, discussing the instructions and questions to be asked during an activity stresses the importance of verbalization while playing with numerical activities with children. As previous studies have shown, parents' verbalization during numerical activities can have an impact on children's numerical competencies (Gunderson & Levine, 2011; Levine et al., 2010).

Another significant part of this study was having participants implement activities with children, video themselves, and then analyze children's knowledge. This type of activity has been used in several programs with teachers (e.g., van Es & Sherin, 2021) and has been shown to increase teachers' ability to notice and attend to noteworthy features of interactions. Likewise, in the current study, when watching one of the videos, participants also noticed what items not to use, such as those that children would rather eat than count. As Sullivan and Mousley (2001) suggested, the dilemmas that participants raised as they planned, implemented, and then reflected on their own engagement with children, provided a way for the instructor to address and highlight the complexity of early numerical competencies. A next step would be to investigate if participants carry out at home what was learned in the course.

As with professional development for teachers (e.g., Tsamir et al., 2014), we believe that the interweaving of mathematics knowledge, tasks, and practice may have additional benefits. First, it conveys an expectation that at least part of what is learnt in theory can be implemented in everyday environments. This is especially relevant for courses and workshops aimed at adults who are not professional teachers. Second, sharing ideas and experiences with others has the potential to create a community where additional ideas and experiences might be exchanged. Third, in discussing practical implementations together, adults may increase their repertoire of activities, as well as their motivation for engaging young children with numerical activities.

7 Conclusion

While this study focused on set comparison, the framework may be used to design workshops for interested adults in other content areas. For example, a geometry workshop might focus on shapes and what makes a triangle a triangle. Studies have shown that many young children identify a rounded "triangle" as a triangle (Tsamir et al., 2008). A workshop that offers adults mathematical knowledge, such as being able to differentiate between critical and non-critical attributes of geometric shapes, can help adults reply appropriately to incorrect identifications.

This study also has limitations. First, the intervention was conducted as part of a university course with only 18 participants. This is quite different from a workshop designed for parents and other adults, who may or may not have a mathematical and educational background like those in this study. Regarding the mathematics, while the content was not advanced mathematics, it might be that participants in this study were familiar with notions such as one-to-one correspondence, from other mathematical contexts, such as functions. Using different strategies to compare sets might also be considered akin to solving a mathematical problem using various strategies. Thus, it might be that participants' experiences boosted their acceptance of those issues. Regarding their educational background, teachers are trained in professional noticing (Mason, 2011; Sherin & van Es, 2009). Thus, it may not be surprising that participants in the course described in such detail what they had observed when they engaged children with numerical activities. These limitations may leave one wondering if such a course would have the same impact on adults who do not have a mathematical and educational background. However, as teachers learn to notice, so too can others learn to notice.

Parents are interested in receiving activities to help promote their children's mathematical growth (Sonnenschein et al., 2021). A course such as the one described in this study can build on parents', and other adults', positive attitudes, offering them not simply activities, but the underlying mathematical and engagement knowledge needed for carrying out effective playful mathematical activities with children. On a final note, we believe that the CAMAE framework could be used to design similar workshops for adults in other contexts, such as working with parents from low-socioeconomic backgrounds, although perhaps with adaptations. Studies such as these may not be transferable as is, but they provide the basis for future research.

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