How lower-secondary pupils approach and perceive understanding mathematics

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Mathematics requires deep thinking. It may be educationally beneficial if pupils are aware of their own shortcomings in understanding. We investigated how pupils of lower-secondary schools in Prague perceive their understanding mathematics, with a particular focus on whether they have distinctive attitudes towards the quality of their understanding. In the first, quantitative stage, a diagnostic test of surface knowledge and a questionnaire about mathematical understanding were developed and administrated. Using a factor analysis, t-tests and methods of descriptive statistics, we created indices of understanding and ascertained that the respondents were often mixing various levels of depth of their understanding mathematics. The quality of a pupil's understanding was also influenced by many latent factors, including strategic approach to learning, volition to remember facts, ability to solve tasks independently, perfectionism, and also, to some extent, the parental view of mathematics. In the second stage of the research, some individual semi-structured interviews were conducted to illustrate and validate the results. The findings of the study highlight the need to raise pupils' awareness of the quality of their mathematical understanding, since it may influence their willingness to deepen their knowledge in mathematics and subsequently their school performance.

Key words:
lower-secondary pupils,
mathematics,
perception of one's own understanding,
deep understanding,
surface understanding,
strategic approach.

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1 Introduction

Understanding mathematics requires deep thinking and understanding, not only of isolated topics but also connections between them. Understanding mathematics is not something which is either present or absent, but rather we deal with the depth[1] and level of pupils' understanding (on depth see, e.g., Skemp, 1991; Star, 2005; Hiebert & Lefevre, 2009; Hejný, 2012, and Sierpinska, 1994 on 'level'). Practising teachers routinely speak about whether pupils understand or are deficient or lacking in their understanding (Novotná & Janda, 2021). The question arises as to what extent the pupils are aware of what it means to understand mathematics. This question is important as the realisation of the insufficiency of one's understanding is a prerequisite for deepening it successfully. If pupils are not aware of diverse levels of understanding, they might not reach higher levels of reasoning, for example, generalization, and can hardly be expected to make an effort to understand mathematics deeply (Hejný & Kuřina, 2009).

However, there seems to be a shortage of studies on pupils' perception of their understanding mathematics. Drawing on well-known dichotomies in understanding (relational/instrumental, procedural/conceptual), we conceptualised the depth of understanding by distinguishing deep/surface understanding, and added a strategic approach to understanding to interconnect the dichotomous nature of the above mentioned. With the help of a questionnaire and a diagnostic test, we determined groups of lower-secondary pupils tending to deep/surface understanding or strategic approach to understanding. We described them in terms of their approach to and perception of mathematics. In the qualitative, complementary, part of our study, we conducted semi-structured interviews with pupils who showed some surface understanding of their knowledge to illuminate their test and questionnaire results, gain more insight and to assess if they were aware of deficiencies in their knowledge in mathematics and if they would like to deepen it.

2 Theoretical framework and literature review

Our study builds on literature related to the quality of understanding mathematics and the concept of self-efficacy and attitudes.

2.1 Quality of pupils' understanding mathematics

Early mathematics education research introduced dichotomies in understanding. Among the most influential ones are Skemp's *relational* and *instrumental understanding* (Skemp, 1991), and Hiebert and Lefevre's

[1] We use the word "depth" advisedly in the meaning "the quality of showing a clear and deep understanding of serious matters".

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procedural and *conceptual knowledge* (Hiebert & Lefevre, 2009) revisited by Star (2005). In the Czech context, Hejný (2012) introduced the Theory of Generic Models which distinguishes between *mechanical* and *non-mechanical knowledge*. What these dichotomies share[2] is that they approach understanding at either a deep or surface level.

Kieran (2013) notes that these dichotomies, often referred to as skills and understanding, are mutually beneficial to each other and are not exclusive, but linked.

> No longer can the two be viewed as separate entities. Nor is it sufficient to argue that conceptual understanding can lead to the meaningful development of procedural knowledge. Rather, the elaboration of procedures has within itself a conceptual component. (ibid., p. 160)

According to Kieran, "procedural skills adapt over time as the conceptual domain to which they are applied is broadened" (ibid., p. 161) and sometimes it is more efficient to solve a problem using an algorithm we know well than to dissipate our energy thinking deeply about the concept.

With respect to Kieran's objections, we will be using Entwistle and Entwistle's framework and speak about *deep* and *surface understanding* (not strictly dichotomous ones) to capture differences in understanding. Entwistle and Entwistle (1992) distinguish breadth and depth of understanding, the latter being more relevant to our study.

> 'Depth' can be taken to describe the extent to which the student is able to provide an explanation which would answer an examination question, or would satisfy an 'expert'. It can be seen to involve both the amount of relevant detail brought together and the strength of the interconnectedness among the component parts. (Entwistle & Entwistle, 1992, p. 16)

In other words, the pupil with deep understanding can see connections in mathematics, knows when and how to use an algorithm and can explain it. The pupil with surface understanding is able to use the algorithm in restricted contexts (an ability we sometimes refer to in this paper as *algorithmic understanding*), may overgeneralize it to inappropriate contexts, and often learns it by heart.

It would be too simplistic to presume that a pupil tends strictly to either deep or surface understanding; he/she might oscillate between them based on the context. For example, if a simple application of an algorithm in similar contexts is sufficient for being successful at school, there is no need for the pupil to make sense of it. This was observed by Entwistle and Entwistle (1992, pp. 17–18): "…the dichotomous distinction between intentions to understand (deep approach), or to reproduce (surface approach), was too limited to cover the range of comments found in our interviews." The authors also mention that more forms of understanding coexist at the same time in the same student. In such a case we refer to a *strategic approach to understanding*[3] (similarly to Mareš's (1976) strategic approach to learning). Since research has shown that the dichotomous view is not always sufficient, in our study, we focus on surface and deep understanding as well as the strategic approach to understanding.

2.2 Attitudes and motivation

The role of motivation in learning is amply documented (e.g., Hejný, 2012; Hannula, 2014; Middleton, 2014; Muis et al., 2015) and pupils' motivation for and attitudes to mathematics[4] is related to their achievement in mathematics (e.g., Hemmings et al., 2011; Ubuz & Aydinyer, 2017) and may affect pupils' willingness to reach deep understanding. For example, Ubuz and Aydinyer (2017) showed a positive significant correlation between striving for understanding and previous achievement in geometry for middle school pupils.

Measuring the cognitive, affective and behavioural components of a pupil's attitude presents a methodological problem. In qualitative research, researchers typically investigate cognitive and affective components by posing open questions to pupils and/or observing their reactions when solving a task (e.g., Vorhölter et al., 2019). In quantitative research, questionnaires are often used, from which individual statements can be interpreted, a semantic differential used, or indices created (Chvál, 2013). An example is Code et al.'s study (2016) of university students' attitudes towards mathematics. Their questionnaire included 31 statements in 7 categories connected to attitudes: *Growth Mindset, Real World, Confidence, Interest, Persistence, Sense Making* and *Answers*. The category of *Sense Making* concerns students' attitudes towards acquiring deep understanding (e.g., *In math, it is important for me to make sense out of formulas and procedures before I use them*) and we included its relevant elements in our questionnaire.

[2] Without any doubt, there are differences in these theories, more thoroughly described in Novotná (2020).

[3] Although Entwistle and Entwistle do not give a name to this approach, they use the word "strategic" as well.

[4] Mathematics is one of the least favourite and most difficult school subjects in pupils' eyes (e.g., Hrabal & Pavelková, 2010; Chvál, 2013).

2.3 Some intervening factors

Factors that influence whether pupils tend towards deep understanding or remain at the surface level include their teacher's personality and teaching style, the demands of the curriculum and the pupils' background and disposition. It would be impossible for a single questionnaire to capture the factors in their entirety, thus, we limited ourselves to self-efficacy and metacognition as the factors connected to and influenced by pupils' views and to the parental influence. These factors influenced the development of our research tools and play an important role in a pupil's approach to learning in mathematics, as described in the following sections.

2.3.1 Self-efficacy

Self-efficacy is defined as "people's beliefs about their capabilities to produce designated levels of performance that exercise influence over events that affect their lives" (Bandura, 1994, p. 2). It influences how we perceive (mathematics), how we think, how we are motivated and how we behave. It differs between areas (e.g., languages vs mathematics, but also areas within mathematics) and changes through time.

A pupil's ability to evaluate his/her performance has attracted much research attention. Pajares and Graham (1999) showed that self-efficacy was a strong predictor of pupils' performance. Yurt (2014) revealed high- and medium-level significant relations between mathematics self-efficacy and mathematics achievement among 7th graders. Schöber et al. (2018) found that mathematics self-efficacy has a positive effects on later mathematics achievement and vice versa. Smetáčková (2018) demonstrated that self-evaluation and self-efficacy are stronger predictors of actual performance than an external evaluation and ascertained a slight tendency of Czech 8$^{\text{th}}$ grade pupils to overvalue their own achievement in mathematics.

2.3.2 Metacognition

While cognition is involved in doing (in solving mathematical tasks), metacognition "is involved in choosing and planning what to do and monitoring what is being done" (Garofalo & Lester, 1985, p. 164). Metacognitively aware pupils seem to be more effective learners and pupils' "metacognitive awareness is related to their task motivation and their subsequent use of strategies in preparing for classroom assessment" (Hammann & Stevens, 1998, p. 4).

To study such a hidden feature of pupils' thinking, research on metacognition uses questionnaires, interviews, observations or their combinations (e.g., Zimmerman, 2018). Pupils' predictions of their success in solving mathematics problems are often compared with their actual results.

2.3.3 Parental influence

Positive parental attitudes towards mathematics affect pupils' attitudes and willingness to study it (e.g., Bartley & Ingram, 2018; Eccles & Wigfield, 2002). Bartley and Ingram (2018) found a significant positive correlation between parents' mathematical self-efficacy and their children's (age 12–13) interest in mathematics. Tocci and Engelhard (1991) found parental support to be a significant predictor of 13-year-old pupils' attitudes towards mathematics. Hong et al.'s (2010) study supported these findings but did not confirm causality between parents' attitudes and pupils' achievements.

2.4 Research questions

In our study, we investigate pupils' intentions of learning and processing subject matter. We do not deal thoroughly with their beliefs about their capabilities (self-efficacy) or their capability of planning and monitoring what is being done (metacognition), despite the undeniable influence. Therefore, we examine the depth of their understanding and knowledge as explained above. To the best of our knowledge, no study has focused on pupils' perception of the depth of their knowledge in mathematics.

The research is of a mixed design. In the quantitative part, we ask if it is possible to distinguish types of lower-secondary pupils who tend to deep or surface understanding, and/or if they have a strategic approach to understanding (RQ1). We also explore the characteristics of these types of understanding in terms of the pupils' approach to and perception of mathematics (RQ2).

In the qualitative part, five pupils were selected to participate in interviews to identify if their approach to understanding corresponds to the appropriate type. For these pupils, their diagnostic test results indicated some level of surface knowledge but their questionnaire results showed different levels and combinations of surface and deep knowledge and strategic approach to understanding.

3 Methodology

We use John Creswell's explanatory mixed methods design for a stronger methodological framework (Ivankova et al., 2006). An explanatory sequential mixed methods design was used, which involved collecting quantitative data first and illustrating the quantitative results with the help of in-depth qualitative data. In the first, quantitative phase of the study, questionnaire and diagnostic test data were collected from 318 lower-secondary pupils to test their notion of their own understanding mathematics and to assess whether it relates to other variables. The complementary, qualitative phase was conducted to gain more insight into the quantitative results; namely, to explore the perception of understanding of selected participants based on their results in the quantitative phase.

3.1 Sample and research procedure

Four lower-secondary schools in Prague[5] participated in the study in 2019/2020; one class in each grade 6 to 9 was selected. The percentage of children from abroad in these classes did not exceed the average for the Czech Republic (about 2%). The classes were average in relation to pupils' behaviour, school results and learning disabilities.[6]

In two mathematics lessons, the author assigned a questionnaire on the pupils' perception of understanding mathematics and a diagnostic test[7] on fractions (Tab. 1).[8]

Tab. 1: Respondents for the first questionnaire

	School A	School B	School C	School D	Total
6th grade	20	12	17	19	68
7th grade	19	19	24	14	76
8th grade	16	16	22	31	85
9th grade	19	19	25	26	89
Total	74	66	88	90	318

Based on the questionnaire responses and diagnostic test results (Novotná, 2020), 16 pupils who demonstrated surface understanding were invited to participate in interviews. Five were chosen to illustrate the results in this article (Tab. 10). With three respondents from school A (Adéla, Barbora, Eliška), two interviews took place in person before the schools were closed due to the covid pandemic. The last interview and all three interviews with Ferda and Václav were performed online. Each interview took around 45 minutes.

The questionnaire was assigned just before the pandemic closed schools in Spring 2020. We supposed that online teaching, in which teachers were only virtually present, might influence pupils' attitude towards their understanding and thus prepared a "covid questionnaire" which included most of the items of the original questionnaire. After pilot testing ($N = 20$), it was sent online to the pupils of all the classes except School B, which declined further cooperation. Since the questionnaire was not assigned in person, the number of questionnaires returned was lower (Tab. 2).

Tab. 2: Respondents for the "covid questionnaire"

	School A	**School C**	**School D**	**Total**
6th grade	8	14	8	30
7th grade	14	7	11	32
8th grade	4	13	12	29
9th grade	12	15	15	42
total	38	49	46	133

In this article, we focus on the cognitive and affective aspects of attitudes. The behaviour aspect was targeted in another part of the study focused on tutoring.

[5]Convenient sampling, more thoroughly described in Novotná (2020).
[6]Based on the formal and informal evaluations of the teachers and/or headmasters.
[7]The test was composed of tasks from research on fractions as an area in which pupils often fail (e.g., Kieren, 1976; Behr et al., 1983; Vondrová et al., 2015).
[8]Four pilot studies (see Novotná, 2020) were conducted to prepare the diagnostic tools ($N = 234$).

3.2 The questionnaires

Part of the first questionnaire was inspired by Code et al. (2016), some statements were based on the criteria for mechanical knowledge (Hejný & Kuřina, 2009) and other research from Section 2. Pilot studies led to the specification of statements and addition of new ones. For example, as pupils did not distinguish between understanding mathematics and being able to carry out an algorithm, the appropriate items were added.

The part of the questionnaire which is the basis for this article consists of 24 statements divided into four groups (Appendix A), evaluated on a Likert scale (1 – I absolutely agree, 2 – I rather agree, 3 – I don't know, 4 – I rather disagree, 5 – I absolutely disagree). It contains six statements related to surface understanding (group A[9], e.g., *It is not my aim to understand how a math solving process emerged [e.g., adding fractions with different denominators]*), six statements relating to deep understanding (group D, e.g., *When I'm learning something new in math, it's good to look for connections to something I already know [e.g., fractions and decimals]*), and six statements about the strategic approach to understanding (group S, e.g., *For a test, I only learn individual steps in a sample solution method, if it is enough for passing it*). Six other statements (group O) are paired, asking first about the estimated difficulty in mathematics tests (important for the strategic approach), second, about the parents' views of a pupil's knowledge, and third, about the difference between "I can do" and "I understand" mathematics.

Given the importance of pupils' motivation for and attitudes to mathematics, the questionnaire also included items aimed at revealing pupils' perception of the difficulty, popularity and importance of mathematics, evaluated again on a 5-point Likert scale.

To prevent the influence of the order of items, four versions of the questionnaire were created and distributed evenly among the pupils in each class.[10]

The second, "covid-questionnaire" was based on the first questionnaire and included new items connected to changes caused by remote learning with an original aim to discover if and how pupils' perceptions of their own understanding mathematics were changed by the remote learning. However, it transpired that there were no statistical differences between pupils' results in the first questionnaire and in the covid-questionnaire as for their perception of their understanding mathematics (see Novotná, 2020). Thus, in this article, we use the data from the second questionnaire to enrich the data from the first one.

3.3 Interviews

During the semi-structured interviews with the author, the pupils were asked about their school mathematics lessons (when learning a new topic/practising/revising for a test), their ways of learning mathematics, what they do when they do not understand something in mathematics, what it means for them to understand mathematics, how they check it, etc. Their answers from the questionnaires were verified, too (e.g., a pupil explained why they perceived mathematics as extremely important). The tasks from the diagnostic test were discussed in order to verify whether their solutions were algorithmic (surface). Some tutoring was organised to deepen pupils' understanding (not the focus of this study). During the last session, the pupils were asked to summarize their participation in the study with respect to perceived changes.

3.4 Data analysis

The questionnaire data was transcribed, checked and analysed in MS Excel using methods of descriptive statistics (arithmetic mean, variation, median, frequencies). The Statements were analysed using factor analysis in SPSS (Statistical Package for the Social Sciences) to confirm the four theorised categories and to provide more insights into the results.[11] Based on the results of the factor analysis, three indices of understanding were created, indicating which type of understanding a pupil is inclined to. Their characteristics and development are described in Section 4.1.

As the answers of 117 respondents of the "covid questionnaire" could be linked to their responses in the first questionnaire, their pre- and "pandemic" results were compared. There were no statistically significant differences in terms of their perception of their understanding mathematics (Novotná, 2020), and thus, in this article, we use the data from the two questionnaires to enlarge the sample ($N = 324+133$).

[9] The category is labelled group A standing for "algorithmic" to avoid disambiguity with the "strategic" statements. Three of each group A and D statements are formulated generally and three are formulated in the context of fractions.

[10] The construction of the versions and permutation of the statements was based on set parameters described in (Novotná, 2020).

[11] Pupils' evaluations of the 5-point Likert scale were coded 1–5 (1 being the strongest agreement). For the factor analysis, the evaluations of three group S statements were rotated to be oriented to the surface pole. Group S statements including the three rotated statements are referred to as group S'.

Due to the bigger sample, this inaccuracy causes less distortion than if only the data from the original questionnaire was analysed by the factor analysis.

The interviews were video recorded and the author took field notes. The recordings were watched repeatedly, and relevant passages transcribed (Cohen et al., 2007). The passages' relevancy was determined by the focus of our study, e.g., any mention of how pupils learn mathematics or consider the understanding they possess were relevant. The pupils mostly commented on the perception of their knowledge during the first and last interviews.

Based on our focus, six categories were developed in a deductive way (Mayring, 2015). The passages identified were investigated further to determine units of analysis, consisting of one or several sentences referring to one particular category. Each unit was included in one or two categories (Tab. 3); thus, the categories are not mutually exclusive. The categorization was made by the author and her supervisor checked its plausibility.

Tab. 3: Coding manual with examples

Category	Example
Surface understanding	Adéla (8th grade): *I have forgotten to use the common denominator here...* (why?) *We have been taught it like this.*
Deep understanding	Eliška (9th grade): *I know where to start and what I am solving. That when I have an equation, I don't only put the numbers in it and that's it.*
Strategic approach to understanding	Ferda (7th grade): *It* (regular standard tests) *forces me to prepare myself for the lesson, [...] I know exactly what to learn.*
Learning preferences	Václav (6th grade): *Because I have to think about it, and then I understand it more properly if I think harder about it at the time when I don't understand it much.*
Self-evaluation of understanding	Adéla (8th grade): *I am really dumb with this.*
Attitudes towards learning mathematics and towards types of tasks	Eliška (9th grade): *I think it would be more interesting, if it was something else than a pizza, something less typical.*

4 Results

The sample consists of 451 respondents, 245 girls and 206 boys (Tab. 4). The proportion between girls and boys was rather balanced in all the classes.

Tab. 4: Number of respondents

	School A	**School B**	**School C**	**School D**	**Total**
6th grade	28	12	31	27	98
7th grade	33	19	31	25	108
8th grade	20	16	35	43	114
9th grade	31	19	40	41	131
Total	112	66	137	136	451

4.1 Pupils tending to deep/surface understanding or to a strategic approach to understanding (RQ1 and RQ2)

The reliability of the statements was measured with Cronbach's alpha[12] (Appendix B, with KMO indices). The Statements were tested by the factor analysis to reveal latent variables and reduce their number. The Statements were tested in groups (A, D, S, O) and together. No minimum number of factors was set for testing the evaluations of group A and D statements. Four identified factors were labelled according to the prevailing character of the statements in them (Tab. 5). The KMO index is 0.709. The factors explain 52.3% of the total variance, which can be considered a standard result (e.g., Chvál, 2013). The factor "Quality of A and D understanding" explained only 20.7% of the total variance. The other three factors impact on the data as latent variables which can also influence pupils' understanding. However, they describe pupils' diverse characteristics, whose analysis is beyond this article.

[12]The lower values below the threshold of 0.7 could be caused by the complex nature of the Statements. But still, the levels of KMO indices are satisfactory, thus, we accompanied the analysis with the analysis of the Group O statements and other items of the questionnaires.

Tab. 5: Measure of representation of group A and D statements in the factors

Statement	**Factor 1 "Quality of A and D understanding"**	**Factor 2 "Volition to remember"**	**Factor 3 "Capability to try to solve independently"**	**Factor 4 "Perfectionism"**
$a1$	0.593	0.202	0.394	0.397
$a2$	0.252	0.568	−0.402	−0.108
$a3$	0.629	0.169	0.030	0.277
$a4$	0.222	0.562	−0.503	0.059
$a5$	0.594	0.319	0.371	−0.340
$a6$	0.536	0.280	−0.046	0.202
$d7$	−0.410	0.470	0.249	−0.052
$d8$	−0.563	0.278	−0.109	−0.124
$d9$	−0.440	0.362	0.138	−0.135
$d10$	−0.386	0.426	0.162	−0.110
$d11$	−0.395	0.374	0.084	0.060
$d12$	−0.074	0.163	0.516	−0.751
Total variance explained	20.7%	13.9%	9.2%	8.5%

Group S' statements were tested by the factor analysis analogously to the above. Two factors were detected, which explained 52.5% of the total variance (Tab. 6). The KMO index is 0.697.

Tab. 6: Measure of representation of group S' statements in the detected factors

Statement	**Factor 1 "Quality of S understanding"**	**Factor 2**
$s13'$	0.664	0.341
$s14$	0.632	−0.212
$s15'$	0.250	0.868
$s16'$	0.599	−0.056
$s17$	0.741	−0.016
$s18$	0.623	−0.423
Total variance explained	33.8%	18.7%

Three indices were constructed based on the values in Tab. 5 and 6 (*index of surface understanding* (i_a)*, index of deep understanding* (i_d), and *index of strategic understanding* (i_s), calculated as weighted averages[13] where the evaluation of each statement is represented to the extent to which it participates in the factor "Quality of understanding". For example, $i_a = \frac{a1 \cdot 0.593 + a2 \cdot 0.252 + a3 \cdot 0.629 + a4 \cdot 0.222 + a5 \cdot 0.594 + a6 \cdot 0.536}{6}$. The lower the value of i_a (and i_s), the more the respondent agrees with group A statements (and group S statements), and vice versa. For the absolute value of index $|i_d|$, the same rules apply; the nearer the value to zero, the more the pupil is inclined to the given type of understanding. Descriptive characteristics of the indices are in Appendix C. The correlation between the indices i_a and i_s is of a medium strength ($r = 0.45$), such as between i_d and i_s ($r = 0.47$), and the correlation between i_a and i_d is weaker ($r = 0.28$).

Figure 1 and the table in Appendix C show that i_a and i_s are distributed rather symmetrically. The values of i_d are approaching zero; there is also a smaller variability in data below the first quartile (Q1). In all the indices, the arithmetic mean is similar to the median. The outlier in the boxplot of i_a belongs to three respondents (R83, R255, R1176) who rather disagree or absolutely disagree with group A statements (values in Q4). Respondent R255 absolutely disagrees with all other statements as well (values in Q4). Respondents R83 and R1176 rather agree with group D statements (values near the Q1 boundary) and absolutely disagree with group S statements (values in Q4). The outliers in the boxplot of i_d belong to respondents R259 (nearer to zero) and R1013. R259 absolutely disagrees with group S statements (i_a

[13] Initially, the indices were constructed as arithmetic means, as is typical in other studies (e.g., Hrabal & Pavelková, 2010). Considering the other latent variables and the diverse loadings (some of them quite low) to which the individual statements are shared in the "Quality of understanding" factor, a more precise approach was sought. We generated a value of the factor for each respondent, which meant that those items which do not constitute the factor also influence this value. The values for the pupils participating in the interviews strongly correlated with the values of the indices based on the weighted average and thus, this approach was used for the whole sample.

around median, i_s in Q4). R1013 tends to strongly agree with group A and S statements (values in Q1). The outlier in the R42's boxplot of i_s shows that he absolutely agrees with all the Statements (values in Q1). These pupils absolutely agree or disagree with at least one other group of Statements.

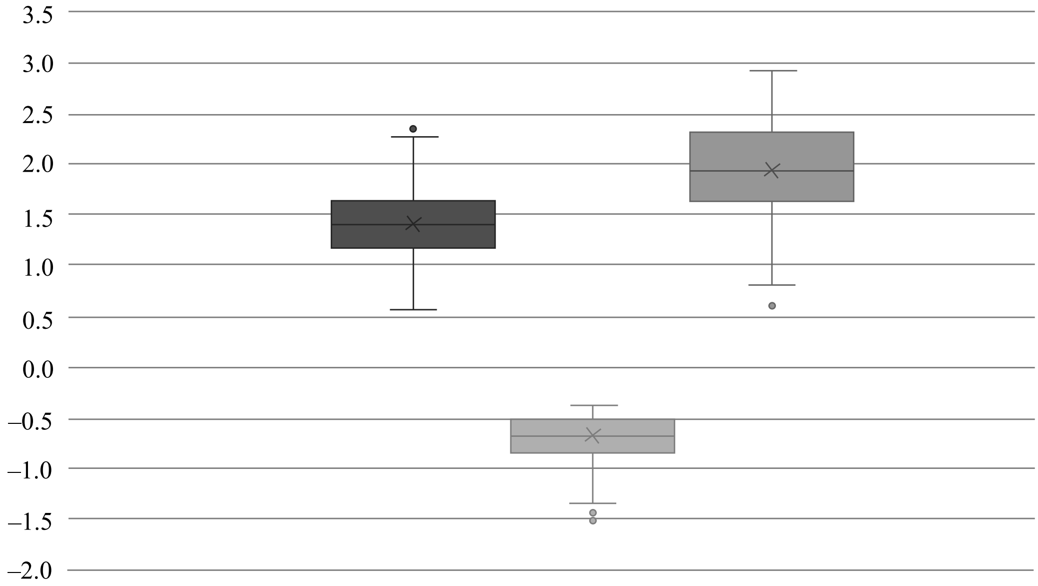

Fig. 1: Boxplots of indices i_a (left), i_d (middle), i_s (right)

The dependency of indices i_a and i_d is in Figure 2; the dashed line, $i_d = 0.21 \cdot i_a - 1.03$, is the trendline. The T value was counted as $T = 0.21 \cdot i_a - i_d - 1.03$ with the standard deviation 0.23. The outliers, farther from the trendline than the distance of one standard deviation (see two lines parallel to the trendline, $N = 144$), were also analysed with respect to other variables. The results of stronger correlations are in Tab. 7.

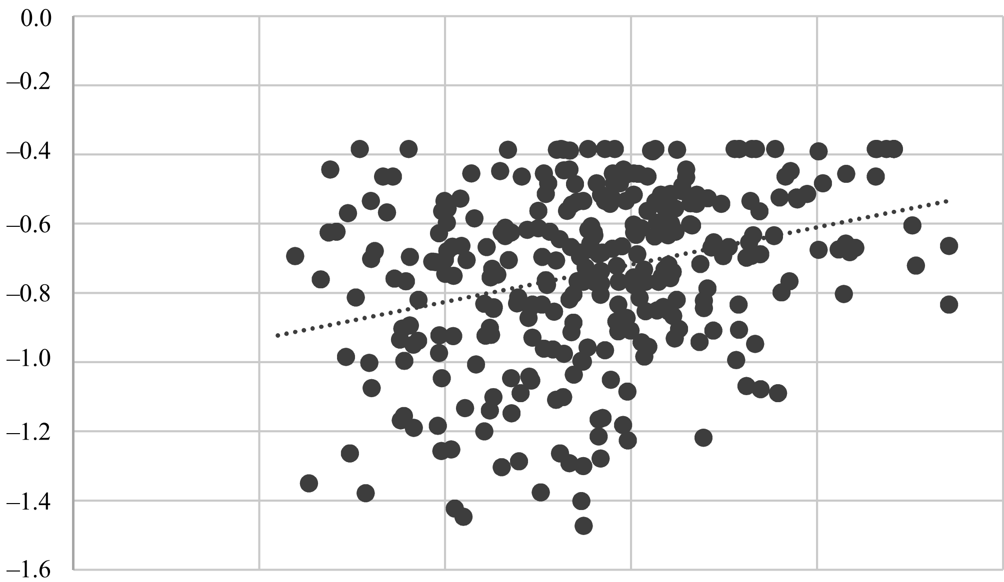

Fig. 2: Dependency of i_a and i_d

Tab. 7: Correlations of T value and other variables

	All T	**Outlaying** T
Variable	r	r
Index i_s	–0.37	–0.37
Statement $o19$	0.22	0.32
Statement $o20$	0.26	0.35
Statement $o22$	0.25	0.43

The pupils' evaluations of group O statements were analysed separately and by the paired t-test for $o19+o20$, $o21+o22$ and $o23+o24$ to see whether each pair was evaluated similarly. The respondents agreed more with every second statement of the pair. While a t-test showed a significant difference between the statements in each pair, an F-test only confirmed it for $o21 + o22$. A small effect size was found for the last two pairs (Tab. 8).

Tab. 8: Results of t-test, F-test and Cohen's d of group O statements

	$o19$	$o20$	$o21$	$o22$	$o23$	$o24$
Arithmetic mean	2.57	2.47	1.88	1.66	2.89	2.43
t-test	$p = 0.0096$**		$p < 0.000$***		$p < 0.000$***	
F-test	$p = 0.908$		$p < 0.000$***		$p = 0.720$	
Cohen's d	0.08		0.23		0.40	

** $p < 0.01$, *** $p < 0.001$

We also analysed how pupils perceived mathematics in respect to its difficulty, popularity and importance, and what their mark in mathematics was at the end of the previous academic year. A weaker significant correlation was found between index i_a and other variables; however, a negative correlation was found between index i_d and other variables (Tab. 9).

Tab. 9: Correlation coefficients between i_a and i_d and other variables

Correlation		r
i_a	Difficulty	0.1818
	Popularity	−0.3140*
	Importance	−0.1920
	Mark	−0.1980
i_d	Difficulty	0.1266
	Popularity	−0.2460*
	Importance	−0.2410*
	Mark	−0.2190*

* weak

The respondents were divided into groups according to their location in one of four quartiles of index i_a, $|i_d|$ and i_s. Almost every combination out of 64 was represented by some respondents (Appendix D). The lower the quartile, the higher agreement with the statements. The least frequent combinations ($N \leq 20$) comprise pupils with a high agreement to group A and D. The most frequent ($N \geq 35$) are combinations with a strong agreement to group A and a strong disagreement with group D, or vice versa. Groups with no stronger preferences towards any of i_a and i_d (between Q2 and Q3 in both) are well-represented (N from 31 to 34); in some of them, i_s is also located in a lower quartile.

No significant differences were found with regard to school grade, gender or type of school in our analysis; therefore, this data is not described in this article.[14]

4.2 Qualitative findings (illustration)

As mentioned above, the interviews were conducted to provide more insight into the quantitative findings. Since the sample is not representative ($N = 5$)[15], we only use the knowledge from the interviews to illustrate the quantitative findings and to show the way of possible future research.

The pupils participating in the interviews were the ones for whom some surface understanding was determined in the diagnostic test. First, the results gained in the interviews were compared with the quantitative results (Tab. 10).

In the interviews, the respondents rarely commented on the depth of their knowledge on their own and if they did, they mostly expressed confusion about why they failed. For example, Barbora who tends towards deep understanding but often resorts to a surface approach said: *I often think I understand it, but then I do not get the correct solution [...] I don't know why.* Or Adéla who strongly tended towards surface understanding during the interviews mentioned: *At the beginning, you must only find the formulas in the textbook [...] and then memorize it and practise.*

[14] However, the data set for each class is rather small and, thus, it would be quite surprising if any statistical significance was found in this aspect.

[15] Our aim was to interview more pupils, however, it was not possible due to the pandemic situation.

Tab. 10: Characteristics of pupils: grade, indices, quartiles and test result (out of 27 points)

Pseudonym	Grade	i_a	Quartile	i_d	Quartile	i_s	Quartile	Test result
Adéla	8	1.06	Q1	−0.78	Q3	1.75	Q2	21p
Barbora	7	1.39	Q2	−0.72	Q3	1.35	Q1	14p
Eliška	9	1.65	Q4	−0.46	Q1	1.72	Q2	20p
Ferda	7	0.80	Q1	−0.66	Q2	1.45	Q1	22p
Václav	6	1.65	Q4	−0.47	Q1	1.32	Q1	26p

Most pupils did not seem to be aware of the surface knowledge they possessed (e.g., Barbora: *I often think I understand it, but the solution is not correct*). An exception is Václav, highly performance motivated, who mentioned hating the feeling when he did not understand something well (deeply): *I am quite nervous that I don't know how to do it, and then I try to solve it in vain, even though I don't know how. It isn't the best feeling.* His statement is in line with the values of indices (i_d and i_s in Q1, i_a in Q4). Václav is the only respondent who indicated that he wanted to get rid of his surface knowledge, e.g., stating: *I am quite nervous, when I don't know how to do* [solve] *it, and then I try to do it, even though I have no idea what I am doing. This is not the best feeling and I am trying to avoid this.*

While for Adéla[16], Barbora, Eliška and Václav, i_a, i_d and i_s agree with the findings from the qualitative part, for Ferda, some puzzling results were found. Ferda is most likely not aware of his surface knowledge. It was clear during the interviews that he tended towards it (moreover, i_a is the lowest, see Tab. 10) and no effort towards deep knowledge was visible. Among others, Ferda claimed: *I guess there is nothing* (in mathematics) *I completely don't know.* He also stressed that when tested at school, he mostly solved all the tasks correctly (but, in his words, they were given easy and standard tasks to which they had been taught a model solution). However, the value of his i_d is in Q2. The i_s (Q1) could be explained by his external motivation, the i_d (around median) might be influenced by the other latent variables, above all "Volition to remember". Ferda also mentioned noticing a change in his attitude during the remote learning*: I am not so good in learning. [...] I did not understand the subject matter so fast, I wasn't motivated, normally I study for tests, so I wasn't revising...*

5 Discussion

In the study, we highlighted the cognitive and affective components of attitudes and uncovered three indices accounting for pupils' attitudes to their understanding mathematics. We found that in our sample, the groups of pupils with strong tendencies towards surface understanding and away from deep understanding or vice versa were most common but not prevalently so. This is in accordance with Entwistle and Entwistle's (1992) findings that mixed types of understanding often coexist in a student. Additional latent variables were shown to intervene, grouped in the factors "Volition to remember", "Capability to try to solve independently" and "Perfectionism". A strategic approach to understanding also influences the perception. Entwistle and Entwistle (1992, p. 27) even state: "To be really successful, it seemed essential that students were strategic" and they also mention the significant role of context. Therefore, their findings support the view that understanding is a multidimensional concept and has to be linked to many other contextual variables (as shown in the qualitative part of this study). Weaker or medium-strong correlations were found between the indices and evaluations of self-reported popularity and importance of mathematics and a school grade.

Regarding the pupils with outlier indices of surface and deep understanding, a medium-strong correlation was found between these values and the statements related to a pupil's perception of his/her understanding ($o19$, $o20$) and those related to parents ($o22$); therefore, the parents' attitudes could influence the pupils' perception. Similar results have been obtained by others (see Section 2.3.3). Oluwatelure and Oloruntegbe (2008) investigated the effects of parental involvement on students' attitudes in biology and chemistry and hypothesise that "a greater academic progress can be achieved by students" if their parents have a good influence on them (ibid., p. 8).

Between the pairs of group O statements, the only statistical difference was found between $o21$ (*it is important for my parents that I have good school grades in mathematics*) and $o22$ (*it is important for my parents that I understand mathematics well*). Pupils might feel some difference between the grades received and their understanding of mathematics; however, our results also show that they do not feel a significant difference between understanding mathematics ($o20$) and being able to do it well ($o19$). As discussed in Section 2.3.2, when not advisedly led to noticing the difference, many pupils are not able to spot it and, thus, this result is not surprising.

[16] Adéla's test result was not so hight due to her imprecise drawings and numerical errors.

In contrast to our expectation, there were no differences in the evaluations of the Statements between pupils with different attitudes to mathematics who perceived its difficulty and importance differently. The popularity of mathematics weakly correlates with i_a and i_d, whereas the importance and also the school grade only correlate weakly with i_d. Thus, we may assume that pupils tending to deep understanding perceive mathematics as more difficult, but they often have better school marks and like mathematics more than pupils tending to surface understanding. More difficult subjects tend to be liked less by pupils, but Hrabal and Pavelková (2010) found that 40% of Czech lower-secondary pupils who liked mathematics perceived it as easy (or very easy) but 25% of them perceived it as (very) difficult. This discrepancy, noticed by the authors but not explained, may be caused by different approaches of pupils towards understanding mathematics.

The interview data showed that either the three index levels, or the test results alone do not present a sufficient indicator to describe a pupil's understanding in all cases. It is better to accompany them by the interpretation of other statements from the questionnaire and discussing them with a pupil. Interviews, therefore, seem to be an important part of investigating pupils' perception of their understanding. We can very tentatively conclude that the three sources of information together provided us with a fitting image of a pupil's understanding, so follow-up interviews may be a promising line of future research.

The interviews revealed that pupils scarcely showed signs of contemplating the depth of their understanding. Entwistle and Entwistle (1992) even found that for more mature students (final year at a university) it is extremely difficult to provide an appropriate account of their understanding. In spite of our efforts, most pupils did not show any indication of being aware of their surface understanding, and/or being disquieted by the depth of their understanding mathematics. We may only speculate why. The most apparent reason seems to be that they are not used to distinguishing different levels of understanding and approaching mathematics at different levels at school, and it is therefore difficult for them to do so. Similarly, Usiskin (2015) pointed out that pupils were often not aware of the multidimensionality of understanding a concept if they were not led by a teacher. Some pupils may not have well-developed metacognition and/or metalanguage to think about and describe their state. This may be a problem, as some argue that "metacognition seems to be one of the most important predictors of mathematical performance" (Desoete & De Craene, 2019, p. 565). Some pupils may also be satisfied with their level of understanding, or may have abandoned the thought they could ever understand mathematics in a deeper way, and accordingly do not want to make an effort. Pajares and Usher (2008) express the same idea and support it by a case study, showing that a teacher's approach can change such pupils' mathematics self-efficacy and make them rethink their attitudes.

6 Limitations, implications and conclusions

The results of our research must be seen in the light of several limitations. The first is the size of the sample. The original plan, which could not be implemented due to the coronavirus restrictions, would have provided us with a bigger sample and enabled us to generalise. The factor analysis would also have stronger validity.

When considering our results, the question arises of whether approaching pupils' understanding in terms of surface/deep understanding and strategic approach to it is expedient. In concordance with Kieran (2013), it was confirmed that a dichotomous view of surface and deep understanding is not sufficient. Despite conducting and evaluating multiple pilot studies, the factor analysis revealed other latent variables that influence the pupils' perception of the quality of their understanding. These factors should be further investigated, separately from those variables we have included. Moreover, factors such as pupils' social environment were investigated only marginally in our study and thus our findings must be taken cautiously. The values of the indices for individual pupils should be validated by results from interviews and mainly by the interpretation of group O statements and other items from the questionnaire about school mathematics (described in Novotná, 2020).

When analysing the data, a problem with a subjective perspective in evaluating Likert-scales by individual respondents might arise. This distortion could be avoided, e.g., by using the anchoring vignette technique (Voňková & Hullegie, 2011). Unfortunately, the vignettes were not included in the questionnaire due to its length, which could have discouraged young respondents.

The results would be more robust if we could compare the indices obtained with the teaching style of the mathematics teacher and especially with his/her attitudes towards the quality of understanding (theirs and also the pupils'). This topic is available for further research.

We are aware of some limitations of the calculation of indices, namely that the extent to which individual statements influence the value of the factor may differ depending on the sample, rotation in the factor analysis, etc.

We believe that we brought insight into the under-researched area of pupils' perception of the depth of their understanding mathematics and that the strength of our study lies in the development of a mixed research methodology. The indices i_a, i_d and i_s together with the referential standards (Appendix C) enable us to analyse pupils' attitudes towards their understanding. It transpired that the depth of pupils' own understanding is influenced by other latent factors, too.

We showed that it is convenient to supplement the quantitative results by an interview. As inferred from the questionnaires and test results data and illustrated with our qualitative analysis, pupils may often not be aware of the depth of their understanding and they mix surface and deep understanding. This has implications for teachers.

Finally, our study is particular to the Czech context and includes a limited number of participants. It remains to be seen whether the results are valid more broadly.

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References

Bandura, A. (1994). Self-efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behavior* (vol. 4, pp. 71–81). Academic Press. (Reprinted in H. Friedman [Ed.], *Encyclopedia of mental health*. Academic Press, 1998.)

Bartley, S. R., & Ingram, N. (2018). Parental modelling of mathematical affect: Self-efficacy and emotional arousal. *Mathematics Education Research Journal*, *30*, 277–297. https://doi.org/10.1007/s13394-017-0233-3

Behr, M., Lesh, R., Post, T., & Silver, E. (1983). Rational number concepts. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91–125). Academic Press.

Chvál, M. (2013). Změna postojů českých žáků k matematice během školní docházky [Change of attitudes of Czech pupils towards mathematics during school attendance]. *Orbis scholae*, *7*(3), 49–71. https://doi.org/10.14712/23363177.2015.13

Code, W., Merchant, S., Maciejewski, W., Thomas, M., & Lo, J. (2016). The mathematics attitudes and perceptions survey: an instrument to assess expert-like views and dispositions among undergraduate mathematics students. *International Journal of Mathematical Education in Science and Technology*, *47*(6), 917–937. https://doi.org/10.1080/0020739X.2015.1133854

Cohen, L., Manion, L., & Morrison, K. (2007). *Research methods in education.* Routledge. https://doi.org/10.4324/9780203029053

Desoete, A., & De Craene, B. (2019). Metacognition and mathematics education: An overview. *ZDM Mathematics Education*, *51*, 565–575. https://doi.org/10.1007/s11858-019-01060-w

Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values and goals. *Annual Review of Psychology*, *53*, 109–132. https://doi.org/10.1146/annurev.psych.53.100901.135153

Entwistle, N., & Entwistle, A. (1992). *Developing, revising, and examining conceptual understanding: the student experience and its implications*. Centre for Research on Learning and Instruction, University of Edinburgh.

Garofalo, J., & Lester, F. (1985). Metacognition, cognitive monitoring and mathematical performance. *Journal for Research in Mathematics Education*, *16*(3), 163–176.

Hammann, L. A., & Stevens, R. J. (1998). Metacognitive awareness assessment in self regulated learning and performance measures in an introductory educational psychology course. *Paper presented at the Annual Meeting of the American Educational Research Association* (San Diego, CA, April 13–17, 1998).

Hannula, M. S. (2014). Affect in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 23–27). Springer.

Hejný, M. (2012). Exploring the cognitive dimension of teaching mathematics through scheme-oriented approach to education. *Orbis scholae*, *6*(2), 41–55. https://doi.org/10.14712/23363177.2015.39

Hejný, M., & Kuřina, F. (2009). *Dítě, škola a matematika: konstruktivistické přístupy k vyučování [Child, school and mathematics: Constructivist approaches to teaching]*. Portál.

Hemmings, B., Grootenboer, P., & Key, R. (2011). Predicting mathematics achievement: The influence of prior achievement and attitudes. *International Journal of Science and Mathematics Education*, 9, 691–705. https://doi.org/10.1007/s10763-010-9224-5

Hiebert, J., & Lefevre, P. (2009). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Routledge.

Hong, S., Yoo, S., You, S., & Wu, C. (2010). The reciprocal relationship between parental involvement and mathematics achievement: Autoregressive cross-lagged modeling. *Journal of Experimental Education*, *78*(4), 419–439. https://doi.org/10.1080/00220970903292926

Hrabal, V., & Pavelková, I. (2010). *Jaký jsem učitel [What kind of a teacher I am]*. Portál.

Ivankova, N. V., Creswell, J. S., & Stick, S. L. (2006). Using mixed-methods sequential explanatory design: From theory to practice. *Field Methods*, *18*(1), 3–20. https://doi.org/10.1177/1525822X05282260

Kieran, C. (2013). The false dichotomy in mathematics education between conceptual understanding and procedural skills: An example from algebra. In Leatham, K. R. (Ed.), *Vital directions for mathematics education research* (pp. 153–171). Springer. https://doi.org/10.1007/978-1-4614-6977-3_7

Kieren, T. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Number and measurement. Papers from a research workshop* (pp. 101–150). Columbus, Ohio.

Mareš, J. (1998). *Styly učení žáků a studentů [Learning styles of pupils and students]*. Portál.

Mayring, P. (2015). Qualitative content analysis: Theoretical background and procedures. In A. Bikner-Ahsbahs, Ch. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education. Examples of methodology and methods* (pp. 365–380). Springer.

Middleton, J. A. (2014). Motivation in mathematics learning. In S. Lerman (Ed.), *Encyclopedia of mathematics education.* Springer.

Muis, K. R., Psaradellis, C., Lajoie, S. P., Di Leo, I., & Chevrier, M. (2015). The role of epistemic emotions in mathematics problem solving. *Contemporary Educational Psychology*, *42*, 172–185. https://doi.org/10.1016/j.cedpsych.2015.06.003

Novotná, G. (2020). *Vnímání kvality vlastního poznání v matematice a jeho souvislost s individuálním doučováním [Perception of the quality of one's own knowledge in mathematics and its connection to private supplementary tutoring]* [Ph.D thesis, Charles University]. http://dspace.cuni.cz/handle/20.500.11956/121355

Novotná, G., & Janda, D. (2021). Jak vybírat úlohy do hodiny a jak sestavit vhodnou taxonomii výukových cílů? [How to choose mathematical tasks and how to use appropriate taxonomy of learning objectives]. In N. Vondrová (Ed.), *Dva dny s didaktikou matematiky 2021. Sborník příspěvků* (pp. 127–130). PedF UK.

Oluwatelure, T. A., & Oloruntegbe, K. (2008). Effects of parental involvement on students' attitude and performance in science. *The Social Sciences*, *3*, 573–577.

Pajares, F., & Graham, L. (1999). Self-efficacy, motivation constructs, and mathematics performance of entering middle school students. *Contemporary Educational Psychology*, *24*(2), 124–139.

Pajares, F., & Usher, E. L. (2008). Self-efficacy, motivation and achievement in school from the perspective of reciprocal determinism. In M. L. Maehr, S. A. Karabenick, & T. C. Urdan (Eds.), *Social Psychological Perspectives* (pp. 391–423). Emerald Group Publishing.

Schöber, C., Schütte, K., Köller, O., McElvany, N., & Gebauer, M. M. (2018). Reciprocal effects between self-efficacy and achievement in mathematics and reading. *Learning and Individual Differences*, *63*, 1–11. https://doi.org/10.1016/j.lindif.2018.01.008

Sierpinska, A. (1994). *Understanding in mathematics.* The Falmer press.

Skemp, R. R. (1991). *Mathematics in the primary school.* Billing & Sons Ltd.

Smetáčková, I. (2018). Obliba školní matematiky a její souvislost s externím hodnocením a sebehodnocením [Preference of school mathematics and its link to external evaluation and self-evaluation]. *Scientia in educatione*, *9*(2), 44–56. https://doi.org/10.14712/18047106.1049

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for Research in Mathematics Education*, *36*(5), 404–411. https://doi.org/10.2307/30034943

Tocci, C., & Engelhard, G. (1991). Achievement, parental support and gender differences in attitudes toward mathematics. *The Journal of Educational Research*, *84*(5), 280–286. https://doi.org/10.1080/00220671.1991.10886028

Ubuz, B., & Aydinyer, Y. (2017). Measuring striving for understanding and learning value of geometry: a validity study. *International Journal of Mathematics Education in Science and Technology*, *48*(7), 1072–1086. https://doi.org/10.1080/0020739X.2017.1327087

Usiskin, Z. (2015). What does it mean to understand some mathematics? In J. C. Sung (Ed.), *The proceedings of the 12th International Congress on Mathematical Education* (pp. 821–842). Springer. https://doi.org/10.1007/978-3-319-12688-3

Vondrová, N., Rendl, M., Havlíčková, R., Hříbková, L., Páchová, A., & Žalská, J. (2015). *Kritická místa matematiky základní školy v řešeních žáků.* [Critical places of lower secondary mathematics in the solutions of pupils]. Nakladatelství Karolinum.

Voňková, H., & Hullegie, P. (2011). Is the anchoring vignette method sensitive to the domain and choice of the vignette? *Journal of the Royal Statistical Society: Series A (Statistics in Society)*, *173*(3), 597–620. https://www.jstor.org/stable/23013513

Vorhölter, K., Krüger, A., & Wendt, L. (2019). Metacognition in mathematical modeling – an Overview. In S. Chamberlin, & B. Sriraman (Eds.), *Affect in mathematical modeling* (pp. 29–51). Springer. https://doi.org/10.1007/978-3-030-04432-9_3

Yurt, E. (2014). The predictive power of self-efficacy sources for mathematics achievement. *Education and Science*, *39*(176), 159–169. https://doi.org/10.15390/EB.2014.3443

Zimmerman, B. J. (2000). Self-efficacy: An essential motive to learn. *Contemporary Educational Psychology*, *25*(1), 82–91. https://doi.org/10.1006/ceps.1999.1016

7 Appendices

Appendix A: The perception of one's own quality of understanding statements (translated from Czech)

Surface

- $a1$. It is not my aim to understand how a math solving process emerged (e.g., adding fractions with different denominators).
- $a2$. When I am able to repeat everything (e.g., about multiplying fractions) after the teacher, it means I understand it well.
- $a3$. When solving math tasks, it is sufficient for me to remember the steps of the process (e.g., how to proceed when subtracting two fractions). I don't care why it works.
- $a4$. I understand math when I can solve a task using a solving process I know from school.
- $a5$. It is not my aim to understand how math formulas were created.
- $a6$. I must remember math formulas and solving processes. When I don't remember some, I can't do anything to figure it out.

Deep

- $d7$. When I'm learning something new in math, it's good to look for connections to something I already know (e.g., fractions and decimals).
- $d8$. My aim is not only to solve a task, but also to understand why I solved it like that (e.g., why I have to multiply two fractions).
- $d9$. I'd like to know fractions so well that I would be able to explain everything about them to someone else.
- $d10$. When I don't understand a math assignment, I don't give up and try to get it.
- $d11$. I feel free to ask my math teacher about something to understand the subject matter better.
- $d12$. It is normal to make a mistake when solving a math task during a lesson.

Strategic

- $s13$. Even though it would be enough for a test to learn a solving process by heart, I still try to think about it and understand it properly.
- s14. For a test I only learn individual steps in a sample solution method, if it is enough for passing it.
- $s15$. I try to understand a solving process well, so that I could modify it, if needed.

- $s16$. I prepare at home for a math lesson as carefully as the teacher requires.
- $s17$. I learn only those things which are necessary to pass a test.
- $s18$. I keep thinking during a math lesson only if it is needed.

Other

- $o19$. I think I can do math well.
- $o20$. I think I understand math well.
- $o21$. It is important for my parents that I have good school grades in mathematics.
- $o22$. It is important for my parents that I understand mathematics well.
- $o23$. Our teacher doesn't give us non-standard tasks in a test.
- $o24$. In a test we often get tasks where it is not enough to remember a solving process by heart.

Appendix B: Reliability (Cronbach's alpha) and KMO indices of the Statements

	Cronbach's alpha	KMO
Group A statements	0.63	0.68
Group D statements	0.54	0.71
Group S' statements	0.64	0.73

Appendix C: Descriptive characteristics of indices i_a, i_d and i_s

	i_a	$\|i_d\|$	i_s
Arithmetic mean	1.43	0.72	1.95
Possible minimum	0.47	0.38	0.59
Achieved minimum	0.55	0.38	0.59
Q1 boundary	1.18	0.52	1.63
Median	1.42	0.68	1.94
Q3 boundary	1.64	0.85	2.31
Achieved maximum	2.36	1.50	2.92
Possible maximum	2.36	1.92	2.92

Appendix D: Distribution of the respondents in quartiles for the indices i_a, i_d and i_s

i_a	i_d	i_s		
Q1	Q4	Q1	25	44
		Q2	15	
		Q3	3	
		Q4	1	
	Q3	Q1	13	29
		Q2	9	
		Q3	6	
		Q4	1	
	Q2	Q1	8	28
		Q2	7	
		Q3	9	
		Q4	4	
	Q1	Q1	4	15
		Q2	0	
		Q3	7	
		Q4	4	

i_a	i_d	i_s		
Q2	Q4	Q1	13	34
		Q2	11	
		Q3	7	
		Q4	2	
	Q3	Q1	9	33
		Q2	12	
		Q3	9	
		Q4	3	
	Q2	Q1	1	21
		Q2	10	
		Q3	7	
		Q4	3	
	Q1	Q1	2	20
		Q2	4	
		Q3	6	
		Q4	8	

i_a	i_d	i_s		
Q3	Q4	Q1	8	23
		Q2	8	
		Q3	7	
		Q4	0	
	Q3	Q1	8	31
		Q2	6	
		Q3	12	
		Q4	5	
	Q2	Q1	7	34
		Q2	9	
		Q3	5	
		Q4	13	
	Q1	Q1	3	29
		Q2	3	
		Q3	6	
		Q4	16	

i_a	i_d	i_s		
Q4	Q4	Q1	4	14
		Q2	5	
		Q3	2	
		Q4	3	
	Q3	Q1	3	16
		Q2	4	
		Q3	8	
		Q4	1	
	Q2	Q1	2	46
		Q2	6	
		Q3	18	
		Q4	20	
	Q1	Q1	1	34
		Q2	1	
		Q3	7	
		Q4	25	