# Secondary and university students' understanding of independence and conditional probability

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The primary objective of the study was to investigate the perceptions of Czech secondary and university students regarding independence in tasks involving multiple repetitions of random events and their understanding of conditional probability. The study employed a sample of 43 students, ranging in age from 15 to 23, who engaged in think-aloud interviews. The selection of eight tasks was based on existing literature. A qualitative analysis of the interview transcripts established that students encountered difficulties comprehending the concepts of independence and conditional probability, irrespective of whether they had previously undertaken a university course on probability. Notably, certain misconceptions about independence only surfaced in more challenging tasks, wherein students relied more on their intuition than their acquired knowledge. The misconceptions primarily manifested when describing the random space. The research findings have significant educational implications, which are discussed.

Key words:
independence,
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intuitive perceptions,
teaching of probability.

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## 1 Introduction

In today's scientific, economic, and technical fields, as well as in society and schools, stochastic and statistical models are of great importance. We encounter them in professional and daily life.

> We all rely on models to interpret everyday experiences. We interpret what we see in terms of mental models constructed on the basis of past experience and education. They are constructs that we use to understand the pattern of our experience. (Biarholomew quoted in Graham, 2006, p. 194)

Researchers agree that it is important that students come to university with the necessary level of understanding of concepts related to uncertainty and randomness (Nemirovsky et al., 2009). However, the importance of stochastic thinking is not always sufficiently appreciated. Studies on teaching probability show that students often have vague ideas about concepts related to uncertainty, randomness, independence and probability when moving from secondary school to university (Albert, 2003; Batanero, 2015; Evans, 2007; Fischbein, 1975; Fischbein & Schnarch, 1997; Konold, 1989). Students' ideas often suffer from misconceptions or misinterpretations of scientific models. Even students after a statistics course and trained statisticians may retain and use invalid intuitions (Díaz et al., 2010).

Modern probability education focuses on concepts that

> played a key role in the history and form the base for modern theory of probability while at the same time people frequently hold incorrect intuitions about their meaning or their application in absence of instruction. (Batanero, 2014, p. 493)

Such concepts include the ideas of random experiment and sample space, addition and multiplication rules, independence and conditional probability, random variable and distribution, combinations and permutations, convergence, sampling and simulation (Batanero, 2014). Similarly, Gal (2005) includes five big ideas as the building blocks of probability literacy: variation, randomness, independence, predictability or uncertainty. Of the important ideas mentioned above, the research presented in this paper will focus on two closely related: independence and conditional probability. Research has shown that even trained professionals sometimes poorly judge independence and conditional probability. For example, they understand conditioning as a causal relationship (Batanero & Sanchéz, 2005).

Misconceptions and misinterpretations in the perception of the independence of random events are well documented. For example, the gambler's bias and the hot hand fallacy (Roney, 2016) are grounded in a student's belief in the internal connection between consecutive events. We often meet fallacies related to independence when looking for all possible outcomes. Students make errors of order or repetition, provide a non-systematic listing of the sample space or make faulty interpretations of diagrams (Jones et al., 2007). Misconceptions in conditional probability are manifested in confusing independence and causality or the fallacy of the time axis, exchanging the role of events, reversing conditions and conditional statements,


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confusing mutually exclusive events and independent events, etc. (Diaz et al., 2010; Batanero & Borovcnik, 2016).

This study is a part of qualitative research on students' comprehension and conceptualisations of probability and statistics. Specifically, the study focuses on gaining insights into understanding independence and conditional probability among Czech secondary and university students.

# 2 Theoretical framework and literature review

A brief historical and epistemological overview of probability will be given first to provide a wider context for students' interpretations. We will focus on two key concepts (independence and conditional probability) that are essential to learning probability and are our study's focus. Next, we will provide a literature review of studies focusing on students' conceptions of independence and conditional probability.

## 2.1 Conceptions of probability

The mathematical description of probability was provided in 1933 by Kolmogorov's axioms, but there are still extensive discussions about its meaning. Probability interpretation is far from resolved (Galavotti, 2017). Two perspectives on the nature and interpretation of probability can be distinguished – epistemological (with subjective and logical interpretations) and ontological (with frequency-based and propensity interpretations).

The epistemological conception of probability describes our relationship to reality. Randomness is only a lack of information, and probability is a tool for quantifying a lack of knowledge. There are two interpretations. The subjective interpretation views probability "as a personal degree of belief" (Batanero, 2015, p. 36) that a person assigns to an event and is based on their willingness to place a bet regarding the event. According to the logical interpretation, the degree of belief is generally accepted by all observers based on logical arguments. Its basic principle is the principle of indifference which assigns equal probability to all elementary events if there is no reason to prefer any of them.

In the ontological conception, randomness is a substantial part of physical reality and is independent of an individual's beliefs or subjective judgment. Part of the ontological perspective is frequency-based probability. It involves looking for the frequency of occurrence of an event when it is repeated many times. The statistical aspect of probability focuses on "objective mathematical rules through data and experiments" (Batanero, 2015, p. 36). The second branch consists of propensity interpretations based on the inclination or tendency of a situation to end with a given outcome.

## 2.2 Independence and conditional probability

This section will define independence and conditional probability.[1]

*Conditional probability* is a measure of the probability of a random event $A$ occurring, given that an event $B$ has already occurred: $P(A|B) \overset{\text{def}}{=} \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$. Bayes' theorem applies: $P(B|A) = $ $= \dfrac{P(A|B) \cdot P(B)}{P(A)}$, $P(A) \neq 0$.

One way to define *independence* is by the relation in which the intersection is converted into a product. We define two random events $A$ and $B$ to be independent if and only if $P(A \cap B) = P(A) \cdot P(B)$. This definition is symmetric, and there is no need to assume a non-zero probability. The second definition is based on conditional probability. We define two random events $A$ and $B$, where $P(B) \neq 0$, to be independent if and only if $P(A) = P(A|B)$. Both definitions are interrelated, as can be seen from the equality $P(A \cap B) = P(A|B) \cdot P(B) = P(A) \cdot P(B)$.

From the educational point of view, the second definition is more illustrative as it points to the essence of independence. The independence of random events $A$ and $B$ means that information about one of them, e.g., $B$, does not affect the probability of event $A$. The probability of event $A$ does not depend on whether event $B$ has occurred.

[1] Independence was first dealt by founders of probability, such as Abraham de Moivre: "Two events are independent, when they have no connection one with the other, and that the happening of one neither forwards nor obstructs the happening of the other." (de Moivre, 1756, p. 6) Independence and conditional probability was thoroughly analysed by Thomas Bayes and completed by Pierre-Simon Laplace (Dale, 1982).

## 2.3 Students' perception of independence

The perception of independence manifests well in the expectation of results in a sequence of the same partial random trial. Understanding of independence is closely related to examples of repeated random events, such as coin flipping or dice tossing, spinning roulette, in the lottery, but also in the birth of a boy or a girl, in anticipation of a fall or rise in the prices of commodities on the stock exchange, etc. Such examples are often reflected in students' erroneous intuitive ideas about independence or conditional probability. Intuition leads them to abandon the idea of independence and use the pattern of past data to predict the next outcome.

Two well-documented misconceptions in this area are the gambler's fallacy and the hot hand fallacy (Roney, 2016), in which the law of large numbers plays a crucial role. It posits that the ratio of the outcomes corresponding to event $A$ and all the outcomes approaches the probability of event $A$. In the long-term repetition of trials, the probability of event $A$ can be approximately replaced by the actual ratio of the outcomes corresponding to $A$ and all the outcomes. However, this does not apply to a few repetitions. The tendency to make false conclusions about the probability based on a few realisations of the trial is called Kahneman's law of small numbers (Tversky & Kahneman, 1971).

While Piaget and Inhelder (1951) supposed that 15-year-old pupils understood the law of large numbers, later studies brought differing results and showed that people of all ages are prone to erroneous decisions even if they learned formal probability theory (Batanero, 2015). For example, the law of small numbers often influences the expectation of repeated trial outcomes. It manifests itself when the probability of an event is also applied to small numbers of its repetitions, regardless of the low reliability of such a conclusion. This mistake is deeply ingrained in gamblers and is often called the gambler's fallacy (bias). It is based on the mistaken belief that if a random event has occurred more frequently than usual, it will occur less frequently in the future. Similarly, the hot hand fallacy (Roney, 2016) is the belief that there is a greater chance of success after a series of successful trials. Graham explains possible reasons for this misconception:

> People sometimes appeal to the "law of averages" to justify their faith in the gambler's fallacy. They may reason that, since all outcomes are equally likely, in the long run, they will come out roughly equal in frequency. However, the next throw is very much in the short run and the coin, dice or roulette wheel has no memory of what went before. (Graham, 2006, p. 58)

The gambler's fallacy is usually explained by misapplications of heuristics regarding random sequences of events. People sometimes behave as if independent events were related. They are internally convinced that long runs of one outcome are unrepresentative and should be unlikely to occur (Tversky & Kahneman, 1971). Other perspectives on the gambler's fallacy emphasise its psychological aspects. For example, a long run of the same results motivates the participants to make certain conclusions and judgments (Roney, 2016). A long run of red numbers in roulette leads to the belief that it must be interrupted at some time. On the other hand, a long run of basket-shooting successes may increase the expectation of further successes with the justification that the player is simply doing well.

## 2.4 Students' perception of conditional probability

Two conceptions of conditional probability can be distinguished. If there is no causal relationship between events $A$ and $B$, we speak about the epistemological understanding of conditional probability. If a causal relationship exists between events $A$ and $B$, conditional probability is understood substantially: Event $A$ results from event $B$. We call such interpretations *ontological* and *causal*. If we perceive conditional probability causally, a problem arises as the direction of inference from cause to effect cannot be reversed. The casual perception of probability can lead to the so-called Humphreys Paradox, which demonstrates that conditional probabilities are formally symmetric, but cause and effect are not symmetrical (Mccurdy, 1996; Humphreys, 1985).

Mixing the conditional probability $P(A|B)$ and the causal relationship between events $A$ and $B$ is common even for university students (Batanero & Sanchéz, 2005; Díaz & de la Fuente, 2007; Díaz et al., 2010). Another frequently identified misconception is a student's expectation that event B should always precede event A (Batanero & Sanchéz, 2005), also called the fallacy of the time axis (Falk, 1989). Referring to international studies, Díaz et al. (2010) identified other misconceptions concerning independence and conditional probability, such as exchanging the events in conditional probability, confusing mutually exclusive events with independent events, etc.

Tarr and Lannin (2005) present an overview of research documenting the above misconceptions of students of different ages, including undergraduates. They distinguished four levels of students' thinking.[2]

[2]Similarly, Jones et al. (2007) propose four levels in the understanding of probability: subjective, transitional, informal quantitative, and numerical. At the highest level, for example, a student can distinguish independent and dependent events.

Level 1 is characterised by a subjective assessment of the situation according to students' beliefs and experiences. Such students can reason about certain or impossible events but ignore numerical data about probability. They consider consecutive events related and have no images of independence and conditional probability. Students with Level 2 thinking recognise that probability changes in some situations but do not assign numerical values to conditional probabilities. They can distinguish which events are related and which are not. Students with Level 3 thinking discern when one event affects another and can distinguish independence between them. They can quantify probability; however, they have shortcomings when considering independence and determining probability. Students with Level 4 thinking can assign numerical probability values to situations and know the conditions under which these hold. They can evaluate the probabilities of consecutive events and distinguish between dependent and independent events.

Mooney et al. (2014) present a synthesised framework of probabilistic thinking. The lowest is the level of prestructural probabilistic thinking. Students intuitively understand randomness and believe that consecutive events are always related. Their thinking is irrelevant, non-mathematical, or personalised. On the level of unistructural probabilistic thinking, students tend to revert to subjective probabilistic thinking, but they already compare probabilities or determine conditional probabilities. On the level of multistructural probabilistic thinking, student thinking is quantitative and proportional. They, for example, recognise changes in probability and independence in without-replacement events and use ratios, counts, probabilities or odds in judging probabilistic situations. Finally, on the level of relational probabilistic thinking, student thinking shows an interconnection of probabilistic ideas. Students, for example, determine probabilities for complex situations, including non-equally likely situations.

### 2.5 Local context and research questions

In the Czech Republic, probability is not part of the Framework Education Programme for Basic Education, which is the main curricular document, abiding for all schools. Probability only comes at the upper secondary school. According to the Framework Education Programme for Secondary General Education (Grammar schools), the pupils are expected to solve real problems with a combinatorial structure, utilise combinatorial analysis methods when calculating probabilities, discuss and critically evaluate statistical information, select and employ appropriate statistical methods and represent data sets graphically. The Framework is not very specific about the concepts which should be targeted: "random events and their probability, probability of the union and intersection of events, independence of events" (FEP, 2007, p. 24). Lessons on probability are usually taught in the final grades in Czech secondary schools. Before that, students can only be expected to have intuitive ideas about independence and conditional probability. Both concepts are present in Czech mathematics textbooks for secondary schools (Mošna, 2022). Independence is usually defined with the help of the definition of conditional probability. Some textbooks also introduce the independence of three events.

Courses on statistics and probability are common at universities, focusing on applying concepts at technical schools or understanding big ideas at the faculties educating mathematics teachers (Mošna, 2022). The international research introduced in the sections above has shown that many misconceptions and intuitive ideas remain even after a formal education on probability. The question arises whether the same applies to Czech students.

Drawing on the studies above, the present study focuses on the following questions:

1. How do secondary and university students perceive independence in tasks requiring multiple repetitions of random events? To what extent are known misconceptions manifested in their considerations?

2. How do secondary and university students perceive conditional probability? Are there signs of epistemological and causal understanding in their considerations?

## 3 Methodology

### 3.1 Sample and research design

The respondents of the study are Czech secondary school and university students. The sample is convenient. Secondary school and university students were invited to participate in the study by the author. They were the students of the universities where he worked (Czech University of Life Sciences in Prague, Faculty of Education of Charles University in Prague) and graduates and students of a Prague secondary school. The sample was complemented by students from the author's social circle. A heterogeneous sample of students who had not undergone the formal teaching of probability and those who had completed such teaching was sought.

Complete anonymity, protection of all personal data and disposal of all documents after the research was guaranteed to the participants. They were informed that the purpose of their solution of tasks was research, and their answers could not adversely affect them. The names mentioned below are fictitious pseudonyms.

A total of 43 participants (18 females, 25 males) participated in the study. They were divided into three groups:

- Group I – secondary school students with no teaching of probability and statistics (aged 14–18), $N = 12$.
- Group II – secondary school or university students who underwent teaching probability and statistics at secondary schools but had not met these topics at the university (aged 18–20), $N = 17$.
- Group III – university students who had taken a course on probability and statistics (aged over 20), $N = 14$.

Given the research aim and questions, a qualitative research paradigm was used. As the aim was to get an insight into students' images of independence and conditional probability while solving tasks, individual task-based interviews were used. It was considered appropriate as such interviews are "intended to elicit in subjects estimates of their existing knowledge, growth in knowledge, and also their representations of particular mathematical ideas, structures, and ways of reasoning" (Maher & Sigley, 2014, p. 579).

The author conducted the interviews. First, the students were asked: "What do you think the independence of two random events means?" Next, they solved tasks on independence and conditional probability and were asked to think aloud while solving them. One by one, the students were presented with the tasks and had time to think about their solutions. The interviewer did not interfere unless the student was stuck. He provided a hint in such a case. He also asked clarifying questions if the student's statement was not clear. The interviews lasted 30 to 45 minutes. They were video-recorded and transcribed to be analysed.

Carefully constructed tasks are key components of the task-based interview (Maher & Sigley, 2014). The author prepared the tasks focusing on students' understanding of independence and conditional probability based on research literature (Batanero & Sanchéz, 2005; Díaz & Batanero, 2009; Konold et al., 1993). The tasks were piloted with 14 students outside the sample to ensure the comprehensibility and unambiguity of their formulations.

## 3.2 Research tool

The research tool consisted of eight tasks. The first six tasks focused on the perception of independence, and the last two on the perception of conditional probability. The coins and dice were supposed to be symmetrical in the tasks. It was presumed that students had an intuitive understanding that with repeated coin tosses, the result of each toss cannot be predicted, heads and tails have the same chance, heads and tails will occur approximately equally often, and the results of heads and tails alternate unsystematically and irregularly.

Next, we present an *a priori* analysis of all the tasks.

> Task 1: Peter flips two ten-crown coins. What is the probability that both coins land on heads? Dusan flips two coins – a ten-crown and a five-crown. What is the probability that both coins land on heads?

The student must decide whether the set of possible outcomes consists of ordered pairs of heads and tails (permutations) or unordered ones (combinations). The toss of two different coins can be divided into two independent tosses of a ten-crown coin and a five-crown one. The probability that the ten-crown coin lands on heads is $P(A_H) = 1/2$, and similarly for the five-crown coin: $P(B_H) = 1/2$. The probability that both coins land on heads is $P(A_H \cap B_H) = P(A_H) \cdot P(B_H) = \frac{1}{4}$. A tree diagram in Figure 1 depicts the situation. The student must realise that the coins behave the same regardless of what is written on them; in other words, they are always distinguishable. That is why Peter and Dusan get the same results.

Tasks 2 and 3 generalise Task 1 to the twofold multiple-outcome trial and the fivefold two-outcome trial. They aim to determine how much task complexity affects the understanding of independence.

> Task 2: We roll two dice. Is the probability of the outcome of five and six equal to the probability of the outcome of two sixes?

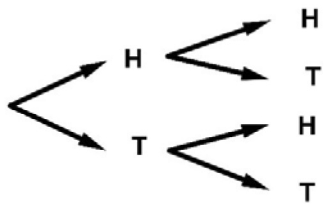


**Fig. 1:** Tree diagram for Task 1

Similarly to Task 1, the set of all outcomes is formed by all ordered pairs of numbers 1 to 6 with repetition. Therefore, the probability of rolling five and six is 1/18 (pairs $[5, 6]$ and $[6, 5]$), and the probability of rolling two sixes is 1/36 (pair $[6, 6]$). The student might intuitively suppose that the probabilities of six and five and two sixes are equal.

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| Task 3: We toss a coin five times. Which of the following outcomes has the lowest probability? Outcomes: a) head, tail, head, tail, head, b) head, head, head, head, head, c) head, tail, tail, head, tail. |

A tree diagram shows that the result of an individual toss is not affected by the previous result. Thus, the probability of all three possibilities is the same, namely 1/32. Answer a) may indicate a student's false belief in the irregular alternation of heads and tails, while answer b) may indicate the gambler's bias.

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| Task 4: We toss a coin five times. Which of the following results has the greatest probability? We will have: a) heads on three coins, tails on the other two coins, b) heads on all five coins, c) heads on two coins, and tails on the other three coins. |

Task 4 is similar to Task 3, but the order of heads and tails does not matter this time. The result in case (b) is 1/32 and in (a) and (c) 5/16. We can use a tree diagram again. To use the solution based on a set of all outcomes, it is necessary to consider an ordered five-tuples of heads and tails as a single outcome.

Successful solutions to Tasks 3 and 4 demonstrate a student's ability to distinguish the specific order of individual outcomes from the total number of outcomes with the same portion of heads and tails.

Tasks 5 and 6 focus on conditional probability and independence. The inductive method (from the specific to the general) is used in Task 5, and the deductive method (from the general to the specific) is used in Task 6. Task 5 falls within the classical interpretation of probability, while Task 6 calls for subjective interpretation.

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| Task 5: We toss a coin repeatedly. The coin has landed on heads ten times in a row. What is the chance that the eleventh outcome is heads again? a) Very small. b) Very large. c) Greater than 1/2. d) Equal to 1/2. e) Less than 1/2. |

Again, as the independence of individual tosses is assumed, the outcome in the eleventh trial is not affected by any of the first ten outcomes. The probability of the outcome "heads in the eleventh tossing" is the same as in any other trial (1/2). It is also important to realise that we use the principle of indifference in the frame of classical (or logical) interpretation of probability. The symmetry of the coin is given.

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| Task 6: Peter and Dusan play table tennis and Dusan has won ten times in a row. What is the chance that Dusan will also win in the eleventh match? a) Very small. b) Very large. c) Greater than 1/2. d) Equal to 1/2. e) Less than 1/2. |

The probabilities of Peter's or Dusan's win need not be equal, and they can even change. We can estimate the next result based on our previous experience. Dusan can probably play table tennis much better than Peter (option b) or c)). Moreover, Dusan's and Peter's performance can vary in each game; they can learn from their mistakes and use the opponent's weak sides in the next match.

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| Task 7: Out of a box with four balls, two black and two white, Dusan draws one ball and puts it in his pocket, and then Peter similarly draws one ball and puts it in his pocket. What is the probability that Dusan has a black ball in his pocket? What is the probability that Peter has a black ball in his pocket? How does the probability that Petr has a black ball in his pocket change when Dusan shows that he has a black ball in his pocket? What is the probability that Dusan had a black ball in his pocket when Peter drew a black one? |

The answers to the first two questions are the same (1/2), even though Dusan draws first. Students may not realise that and may say they cannot answer because they do not know what Dusan has drawn. They might conclude that the probability is 1/3 if Dusan has drawn a black ball and 2/3 if Dusan has drawn a white ball. Such an answer would suggest a causal understanding of probability. If, on the other hand, the student accepts an epistemological explanation, they can use the total probability theorem[3] or a tree diagram to solve the task.

The answer to the third question is easy (1/3). Students might be surprised that the fourth question has the same result. If they understand conditional probability causally, they will conclude that Dusan drew the black ball with probability 1/2, regardless of Peter's following outcome. If they understand it epistemologically, they can determine the correct probability of 1/3.

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| Task 8: There are two ordinary dice in the box (numbered 1, 2, 3, 4, 5, 6) and one special dice where five is replaced by six (numbered 1, 2, 3, 4, 6, 6). Dusan draws one of these dice at random. He throws it. What is the probability that a six will be thrown? Peter does not see the whole process. He thinks about what kind of dice Dusan has probably thrown. What is the probability that Dusan has thrown a special type of dice?<br>Dusan reports rolling a six. What is the probability (for Peter) that Dusan has rolled a special dice? How would the situation change if Dusan reported that he rolled a five? |

This task is similar to Task 7. The total probability theorem yields the probability that Dusan rolls a six: $P(A) = 1/6 \cdot 2/3 + 2/6 \cdot 1/3 = 2/9$. Some authors suggest performing Bayesian calculations with absolute frequencies (Martignon & Wassner, 2002): there are four sixes out of 18 sides (the total number on all three dice), thus $P(A) = 2/9$. The probability that Dusan drew the special dice is $P(B_2) = 1/3$. To calculate how this probability changes after additional information that Dusan has rolled a six, we can use Bayes' formula (see Section 2.2) $P(B_2|A) = \frac{\frac{1}{3}\cdot\frac{1}{3}}{\frac{2}{9}} = \frac{1}{2}$ or the definition $P(B_2|A) = \frac{P(A \cap B_2)}{P(A)}$, $P(B_2|A) = \frac{\frac{2}{18}}{\frac{2}{9}} = \frac{1}{2}$. A simple consideration also leads to a solution: two of the four possible sixes are on the special dice, and two are on the ordinary dice (one six on each of the two ordinary dice).

The last question of Task 8 can uncover the epistemological nature of probability. The probability that Dusan had drawn the special dice when he rolled a five can be found by simple reasoning. This probability is 0 because five is not on the special dice.

## 3.3 Data analysis

The analysis of the interview transcripts was supplemented by two additional sources of data: the students' artefacts, specifically their solutions to the tasks, and the field notes taken by the interviewer (author) during the interviews.

The accuracy of the students' answers for each task and individual was tracked during the initial analysis. Subsequently, a second round of analysis was conducted, wherein the data were coded using a preliminary coding framework developed based on an a priori analysis of the tasks in the research tool. Through iterative data readings, new codes emerged. This process led to establishing a coding framework comprising codes and their corresponding descriptions, drawing on the methodology outlined by Saldaña (2015). The codes encompassed the anticipated misconceptions, incorrect reasoning, and expected justifications for the conclusions drawn by the students. In the subsequent stage, the codes were further categorised into two themes, independence and conditional probability, aligning with the research questions posed in the study.

**Independence:**

Correct solution: Graphs, Sample spaces, Variations, Combinations
Incorrect solution: Faulty interpretation of graph, Faulty sample spaces (Errors of repetition, Errors of order), Incorrect use of variations, Incorrect use of combinations
Misunderstanding the question

Conditional probability:

Correct solution: Conditional probability, Total number of cases
Incorrect solution: Exchanging the events, Fallacy of the time axis, Non-adequate notation
Misunderstanding the question
Interpretation of probability: Rather epistemic, Rather casual

[3] Let us suppose that the set of all outcomes ? can be factorized into two parts $B_1$ and $B_2$ which cover it ($B_1 \cup B_2 = \Omega$), which are disjoint ($B_1 \cap B_2 = \emptyset$), and moreover $P(B_1) \neq 0$, $P(B_2) \neq 0$. Then for random event $A \subset \Omega$, $P(A) \neq 0$, it holds $P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2)$.

For example, the statement in which the student would not answer the question of which dice was originally drawn if a six subsequently fell in Task 8 was coded '**Conditional probability**: Casual understanding, Fallacy of the time axis'.

The author coded the data in two rounds, and the coding consistency was checked by a collaborator in about 20% of the data. Next, the frequency of codes was calculated for Group I, Group II and Group III.

# 4 Results

The results will be presented for each research question separately. They will be illustrated by the students' quotes from the interviews.

## 4.1 Perception of independence

First, we will look at the number of correct answers in each group (Tab. 1). There seems to be a slight improvement in understanding independence with the level of study. However, the sample is small, and we cannot make any conclusions about differences between groups regarding the success rate. Thus, we will present the results for the whole sample.

**Tab. 1:** Absolute and relative frequencies of correct answers for Tasks 1–6 per group

| | Group I ($N = 12$) | | Group II ($N = 17$) | | Group III ($N = 14$) | | In total ($N = 43$) | |
|---|---|---|---|---|---|---|---|---|
| Task 1 | 8 | 66.7% | 15 | 88.2% | 14 | 100.0% | 37 | 86.0% |
| Task 2 | 3 | 25.0% | 5 | 29.4% | 4 | 28.6% | 12 | 27.9% |
| Task 3 | 1 | 8.3% | 4 | 23.5% | 4 | 28.6% | 9 | 20.9% |
| Task 4 | 2 | 16.7% | 6 | 35.3% | 9 | 64.3% | 17 | 39.5% |
| Task 5 | 4 | 33.3% | 12 | 70.6% | 12 | 85.7% | 28 | 65.1% |
| Task 6 | 7 | 58.3% | 16 | 94.1% | 14 | 100.0% | 37 | 86.0% |

In the interviews, students characterised independence as a relationship when two events do not influence each other. They said, for example, that independence "is a state in which the subject is completely unaffected by another", "is when the elements do not interact with each other", or "is the absence of a relationship between two variables". Some students provided a general answer, saying that independence is "freedom", "self-sufficiency", or "non-dependence".

Next, we will present results related to the tasks used in our study.

In Task 1, we focused on the agreement or difference between two situations: tossing two identical and two different coins. Most respondents (37 out of 43) considered it the same. Their arguments were "both cases are identical, coins as coins" or "the value of the coins does not matter". However, only 32 of them could calculate the probability correctly. Jirka's approach (Group I) is worth attention. He initially concluded: "The probability is 1/4 if we toss ten-crown and five-crown coins, and the probability is 1/3 if both coins are ten-crown ones." Then, without the interviewer's prompt, he changed his mind and added to himself: "But no two coins are identical." After this the interviewer initiated a conversation about how the coins could be different. He told Jirka that coins could differ in colour and attempted to lead him to a paradoxical situation where coins would behave differently for a colour-blind observer and a person with normal colour vision. In the end, Jirka insisted that "it depends on the assignment; if the coins are different, the correct probability is 1/4, and if we guarantee they are not different, then 1/3". This indicates Level 1 in Tarr and Lannin's (2005) model.

Mathematically, the solution of Task 2 is similar to that of Task 1, yet many students did not get the correct result. Less than a third of them (12 out of 43) realised that the probabilities of rolling a six and a five differ from the probability of rolling two sixes. In addition, some students gave the correct answer (that the probabilities are different) but based on misconceptions, e.g., that "five rolls more often". A slight complication of the task thus brought completely incorrect considerations.

Task 3 is a generalisation of Task 1 from the mathematical point of view, yet, its solution presented a problem for most students. The correct answer (the probability is the same in all cases) was provided by a few of them (9 out of 43). Outcome b) was considered the least probable, which was expected.

In Task 4, many students realised the connection with the previous task. They found that the total number of tails and heads differed from only one individual outcome (with the same ratio). Alena (Group III) formulated it well: "It looks the same as the last task, but it is not. There seems to be an omission of arrangement, which is a very important aspect of probability." The interviewer confirmed her answer. There were more correct answers than in the previous task, but not by many. Few respondents (17 out of 43) realised that the probability of results a) and c) is greater than the probability in b). Only nine respondents considered a) the most probable, and 12 selected c). No student opted for b). Five out of 43 respondents considered all three results equally probable.

Task 5 asks for the probability that after ten heads, the next outcome will be heads again. 28 students provided the correct variant d). However, it also means that more than a third of students in the sample do not perceive the independence of individual outcomes. Alena (Group III) justified her decision for a) 'Very small': "[The probability] is small, but the chance is a jerk." Another student supported his option e) 'Less than 1/2' by commenting: "…it would be very regular."

Task 6 describes Dusan's chance of winning in the eleventh game after winning the ten previous ones. Most students realised that Dusan is probably a better player and chose b) 'A high probability' (29 out of 43 respondents) or c) 'Greater than 1/2' (8 out of 43 respondents). Contrary to Task 5, a subjective interpretation of probability is needed. An example of students' reasoning is: "It's about [Dusan's] abilities." The subjective conception was also reflected in the statements "[Am I] a bookmaker?" or "I support Peter". Only 6 out of 43 students used the principle of independence and indifference and answered d) 'Equal to 1/2'. Their answers correspond to a subjective perception of probability based on the individual's willingness to bet and may be different for everyone. Peter and Dusan's matches do not have to be independent. They are interconnected and influenced by both players' form and game development.

### 4.2 Perception of conditional probability

The last two tasks focused on students' perception of conditional probability. Table 2 shows that indications of epistemological and causal interpretations appeared in all the groups and that, for many students, it was impossible to decide on the type of interpretation.

**Tab. 2:** Numbers of interpretations of conditional probability in Tasks 7 and 8

| | Group I | Group II | Group III |
|---|---|---|---|
| Indications of epistemological interpretation | 1 | 4 | 5 |
| Indications of casual interpretation | 3 | 8 | 6 |
| It was not possible to decide | 8 | 5 | 3 |

The students' answers to Task 7 fully confirmed the expectation of the causal interpretation. Almost all answered the first question (what is the probability that Dusan will draw a black ball) correctly (40 out of 43). The second question (what is the probability that Peter will draw a black ball) was only answered by 30 students. They missed information about what Dusan had drawn. Alena (Group III) expressed her helplessness: "Am I a clairvoyant?" Some students reached the results of 1/3 when Dusan drew a black ball and 2/3 when he drew a white ball; however, they did not take into account the overall probability of 1/2. Such students answered 1/3 to the third question (how the probability that Peter drew a black ball changes when we know that Dusan drew a black ball). In total, 23 students correctly answered the overall probability of 1/2.

On the contrary, 7 students claimed that the probability that Peter would draw a black ball was the same, whether we know what Dusan had drawn. These students probably understand probability causally and not epistemologically.

The difference between the epistemological and causal conceptions was manifested in the fourth question of Task 7. The relationship between what Dusan drew and what Petr drew can be understood causally or epistemologically. Alena (Group III) stated: "I am slowly losing myself. If Dusan drew first, what Peter drew is irrelevant, right?" She did not realise the relationship between the two events, which indicates Level 2 in Tarr and Lannin's (2005) model. Less than a third (14) of students gave the correct answer.

Task 8 was difficult for the study sample, as only five students answered all the questions correctly. Others often said that the task was beyond their ability and imagination: "I'm at my wits' end", "I don't know anymore", and the like. Most students perceived the epistemological nature of probability as problematic: "The situation is different only psychologically; the probability is the same." (Pavel, Group III)

## 5 Discussion

The study confirmed students' problems with understanding the concept of independence as known from the literature (Díaz et al., 2010; Roney, 2016). In the tasks on independence (1–6), to an extent, the students could create sample spaces for two-stage experiments, which indicates that their probabilistic thinking is on the unistructural level (Mooney et al., 2014). However, their arguments were often incorrect, and they had great difficulties if the tasks got slightly more complicated. Thus, the students cannot construct such sample spaces systematically and have not reached higher levels of probabilistic thinking (multistructural or relational, ibid.).

Students' somewhat vague ideas about independence mainly manifested themselves in tasks related to repeating partial random events. Incorrect conclusions included assigning a higher probability to randomly distributed irregular $k$-tuples of outcomes and a lower probability to $k$-tuples where partial events were repeated or were periodic. The students often missed the principle that, in the case of independence, the probability of one event is not affected by the results of the previous (or subsequent) event.

The study focused on how students' lack of understanding of independence affects the method of estimation and probability calculations in specific cases with repeating trials. The understanding of independence and multiple repetitions of a random event has been addressed by several studies (Batanero & Sanchéz, 2005; Konold et al., 1993). They conclude that students can successfully solve simple tasks, such as repeating a random experiment twice with two outcomes (a double toss of a coin or a toss of two coins). The study presented in the paper showed that problems mainly occurred in more complicated tasks. The students realised that the outcomes of two simple trials were independent and correctly calculated their probability. However, if the task got slightly more complicated, misconceptions about independence occurred, and students could not calculate the correct probabilities. For example, a double experiment of an event with multiple outcomes (rolling two dice) or multiple repetitions of a simple event (tossing five coins) were shown to be problematic tasks for students.

We also showed that the incorrect understanding of the repeated results is often related to the incorrect understanding of the assignment. There is an important difference between a certain sequence (or 5-tuple) of heads and tails with their order on the one hand and a group of outcomes with a certain total number of heads and tails on the other hand. The discrepancy between intuitive perception and calculations seems to be related to the formulation of the question and the answer we are looking for. For example, Task 5 asks for the probability of tossing heads in the eleventh trial, i.e., the probability of tossing heads in one trial after tossing heads in the previous ten rolls. It means something other than the probability that heads will occur in all eleven tosses.[4] Many students found it impossible to accept that the probability of heads is still $1/2$ after ten heads. The intuition of randomness is more suited to a certain irregularity and rotation of partial outcomes with an approximate ratio of $1 : 1$. On the contrary, in a similar case (in Task 6), the students realised that the probability that Dusan would win again after ten previous wins was relatively high because we judge it based on a subjective conception of probability and use an inductive way of reasoning.

In the conditional probability tasks (7, 8), the students could calculate the conditional probability of the events that followed the condition. It indicates an unistructural level of probabilistic thinking (Mooney et al., 2014). Nevertheless, they could not calculate the conditional probabilities when the event had pre-conditions. They could not recognise this situation change, and, thus, their probabilistic thinking remained on the unistructural level.

Several reasons for the incorrect perception of conditional probability are usually discussed in the literature. First, it is a causal perception of conditional probability, its temporal perception, the confusion of considered events and other misconceptions (Díaz & de la Fuente, 2007; Díaz & Batanero, 2009; Tanujaya et al., 2018). In this study sample, conditional probability was mostly perceived as causal. However, epistemological ideas are not related to random events but to information about them. Information about what happened before can be influenced by information about what followed. A two-way epistemological relationship can replace the one-way causal relationship in these considerations. This may lead to some inconsistencies. Cause and effect cannot be confused. The formula for calculating conditional probability can be applied to both sides, i.e., for $P(A|B)$ and for $P(B|A)$ but must be understood epistemologically. In the presented study, the perception of conditional probability based on the position on the timeline was evident. This phenomenon has a notable influence on the causal perception of conditional probability.

Despite the limited sample size and the relatively small number of students within each group, the findings suggest that misconceptions were present in the solutions provided by students across different groups, regardless of their prior exposure to probability education. These findings align with international research, highlighting the propensity for misunderstanding and reliance on intuitive judgments among even experienced professionals (Batanero & Sanchéz, 2005; Díaz et al., 2010).

However, it is important to acknowledge the limitations of our study. Firstly, the small sample size necessitates caution in generalising the results. They should be interpreted in the context of a restricted sample. While we cannot make broad claims, the study does offer insights into how students approach the concepts of independence and conditional probability and how their intuition can lead to misconceptions. A larger sample would be required to validate the uncovered phenomena.

[4] In the former case, it is $1/2$, and in the latter $1/2048$. The former consists of the situation with one toss, and the latter concerns the situation with 11 tosses. Limit theorems can be applied in the latter case but not in the former.

The selection of tasks employed in this study introduces another limitation. Different tasks might provide a more nuanced understanding of students' reasoning in this domain. Further research examining students' perceptions of independence, conditional probability, and probability in general would be beneficial in expanding the knowledge in this area.

## 6 Conclusions and implications

The findings of this study confirmed that students, including those who had undergone a university course on probability, encountered difficulties in comprehending the concepts of independence and conditional probability. Notably, misconceptions regarding independence became apparent in more challenging tasks, wherein students relied more on their intuition than their acquired knowledge. Specifically, students exhibited confusion between scenarios involving the total number of events and those involving a specific sequence of outcomes, including their order. The misconceptions in perceiving independence were particularly evident when describing the random space, encompassing the complete set of outcomes in a random event.

The study has some educational implications. Firstly, it is advisable to solidify students' foundational understanding of independence through ample elementary examples, gradually progressing to tasks of increasing difficulty. When a student encounters difficulty with a complex problem, it can be beneficial to compare the results obtained in easier tasks and those in more challenging ones. Secondly, students should be encouraged to analyse the problem at hand from different angles. For instance, when considering the tossing of five coins, highlighting the distinction between two types of questions can be helpful: one concerning the probability of a specific sequence (or quintuple) of heads and tails in a given order, and another about the probability of a group of outcomes with a predetermined total count of heads and tails. Thirdly, the principles for solving problems in classical probability rely on symmetry. One can extend these principles by utilising symmetry and independence to encompass scenarios involving double or multiple throws or moves. It is often advantageous to decompose random experiments into individual steps. Once the correct calculation methodology becomes sufficiently clear, concepts such as permutations or combinations and the application of combinatorial formulas can be introduced. Experimental verification can further reinforce the obtained results. Lastly, graphical representations, such as tree diagrams, proved effective in delineating the decomposition above into simple partial steps as they illustrate fundamental probability concepts (Díaz et al., 2010; Graham, 2006; Rolka & Bulmer, 2005).

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