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Výčet všech možných případů: Analýza žákovských prací

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Abstrakt

Pokud žákům při výuce pravděpodobnosti nabídneme pestrý výběr kontextů, umožníme jim, aby se učili usuzovat v oblasti ‚neurčitost', což může přispívat k rozvoji jejich kritického myšlení. Článek předkládá studii o tom, jak žáci čtvrtého ročníku rozvíjejí v kombinatorice strategie vyjmenovávání pomocí kritického myšlení. Žáky vyučoval student učitelství. Analýza práce žáků ukázala, jaké postupy používali. To znamená, že prezentované výsledky pocházejí ze sekundární analýzy. Ukazují, že mají-li se u žáků rozvíjet důmyslnější strategie pravděpodobnostního usuzování, musí jim k tomu být poskytnuto dostatek příležitostí. Studie zároveň ukázala, že kritické myšlení je pro rozvoj takového usuzování na jedné straně nezbytné a na straně druhé se jím též rozvíjí.

Klíčová slova: kritické myšlení, strategie vyjmenovávání, pravděpodobnostní usuzování.

Enumerating All Possible Outcomes: An Analysis of Students' Work

Abstract

A variety of contexts in the learning of probability could provide opportunities for students to reason under uncertainty. This kind of reasoning could support students to develop critical thinking practices. This paper presents a study on how children in a grade four classroom developed strategies about enumeration of combinatorics using critical thinking. A preservice teacher taught those students and the analysis of their work showed the procedures they used. Then, these results emerge from a secondary analysis. The results suggest that students need opportunities to develop increasingly sophisticated methods of reasoning probabilistically. This study also suggests that critical thinking is both necessary to develop these modes of reasoning and is developed through this work.

Key words: critical thinking, enumeration strategies, probabilistic reasoning.

1 Introduction

Teaching probability in primary school is now a growing part of the curriculum in the United States and in Canada. The importance of this concept is reflected in everyday life, in which many situations require reasoning under uncertainty. Thus, since 2001, theoretical and frequentist probabilities are now taught in grade 1 until grade 6 in the province of Quebec (Ministère de l'Éducation du Québec, 2001), in qualitative and quantitative ways. Furthermore, subjective probability is part of the secondary school level curriculum. Teachers must provide a variety of learning contexts in order for students to develop conceptual understanding of probabilistic concepts. These contexts must also provide opportunities for students to reason under uncertainty and develop critical thinking practices, such as using criteria when assess an information (Lipman, 2003).

2 Learning about Probability

Learning about theoretical probability involves identifying all the possible cases of an event (sample spaces) and might conduct to develop enumeration processes. Because probabilistic reasoning involves taking into consideration the variability of a compound outcome, combinatorics strategies are part of the complexity to reason under uncertainty. Combinatorics refers here at this idea of grouping items or objects using combinations, permutations or arrangements. A combination is a group of items or events, where the order doesn't matter and will not change the group. In fact order matter with arrangement and permutation, where arrangement involves also repetition of the items or events. As English (2005) points out, although combinatorics plays an important role in probabilistic reasoning, there is little research on children's combinatorial reasoning. That being said, the literature presents very important works on it.

2.1 A framework about enumeration

To this point, research by Piaget and Inhelder (1974) on the idea of chance shows that students aged 7 to 14 can distinguish between a certain event and a possible event. Nonetheless, among young learners, a difficulty persists in terms of the procedure used to list all possible cases. In order to find out the combination of different coloured objects, at the first level, they do a successive construction of pairs, without links between them. At the second level, they try to use a systematic method in order to list all of them. They start with a colour and list all pairs systematically, by adding one colour to another one, which is juxtaposition. At the third level, the procedure is based on a multiplicative structure. Thus, they have to repeat the same operation a number of times. The system is linked to each pairs and thus creates an intersection: For example, AB, AC and BC are the intersection of A, B and C.

This challenge to list all possible cases makes it more difficult for them to begin quantifying probabilities. With this limitation in mind, Jones & al. (1999) developed a four-level frame of reference for probabilistic reasoning that is not related to children's developmental stages. Jones & al. (1999) described the reasoning of individuals at the first level as one where they adopt a limited perspective with respect to assessing probabilistic situations as subjective. They cannot identify all the possible results of an event (*sample space*).[1] The reasoning of individuals at Level 2 is described as transitional (*transitional between subjective and naive quantitative reasoning*). Individuals at this level can identify all the possible cases of a single-stage event and sometimes those of a second stage of experimentation. The reasoning of individuals at Level 3 is referred to as informal quantitative (*informal quantitative reasoning*); they use strategies in a more systematic manner to identify the possible results of a first and second stage of experimentation. Finally, the reasoning of individuals at Level 4 is referred to as numerical (*numerical reasoning*); they use strategies systematically to generate the results of an event of an experimentation using numbers to express probabilities.

2.2 Combinatorics tasks and students' thinking

In a longitudinal study on mathematics investigation toward combinatorics, from first grade to high school and beyond, Maher, Powell and Uptegrove (2011) presented the process of how students learn to justify their solutions to combinatorial problems considered challenging for their age. They identified some strategies employed and justified by grade 3 and 4 students that they reused over a period of time to solve some combinatorial problems. Students used different strategies, such as trial and error, pattern recognition and controlling variables. They refined their representations by reflecting upon the task, discussing in pairs about their ideas and sharing their written and mental strategies (Maher, Sran, Yankelewitz, 2011). Although those students should use critical thinking when making arguments about their strategies, the authors did not discuss critical thinking.

Some questions thus arise. What knowledge might contribute to the development of students' enumeration strategies? What classroom activities could be devised to support students so they can develop increasingly sophisticated reasoning? Likewise, in what ways do this knowledge and these reasoning strategies support students' development of critical thinking? In this paper, we address those questions.

In our point of view, in order to develop sophisticated reasoning, the situation proposed to the students must challenge them. Indeed, Vergnaud (1990) suggests that there is one kind of situation that is challenging for students: "the one that places the student in a context where the learner doesn't have all the competencies needed, so that he is forced to take time to reflect on it, to explore, to hesitate, to try and drives him to success or failure" (Vergnaud, 1990, p. 136). Specifically to the context of learning combinatorics, this kind of situation should also allow students time to find patterns, be systematic, explain and justify their solutions, modify their ideas and generalize findings (Maher, Yankelewitz, 2011).

It should be emphasized that the conceptual field of probabilistic structures (Savard, 2010) can be delineated by situations that give meaning to probabilities, which can be examined according to three different approaches:[2] the theoretical approach, the frequential approach and the subjective approach (Briand, 2005; Caron, 2004; Hawkins, Kapadia, 1984; Konold, 1991). Combinatorics is strongly related to

[1] We would like to state our opinion about this assertion. The fact that an individual is unable to identify all the possible cases does not mean that s/he has used a qualitative method. It means instead that this individual has not mastered the quantitative method of understanding the task, such as using the order or not.

[2] These approaches do not represent components of a linear sequence of levels of learning; instead they constitute different approaches toward probabilities, to which we ascribe a character of reference that is, as comprising the set of situations that endow the concept with meaning (Vergnaud, 1990).

this conceptual field. Reasoning about random phenomena needs to be expressed within the framework of specific terminology, such as possible, certain and impossible.

2.3 A conceptual framework

Jones et al. (1999) describe schemes involved in the probabilistic structures. These authors framed the students' schemes in terms of the core concepts underlying a probabilistic reasoning process, as viewed in relation to the task situations provided to students. Furthermore, they consider these core concepts as the building blocks of probabilities, which themselves are viewed as constituting multi-faceted concepts (Scholz, 1991). The core concepts fit into the conceptual field of probabilities in the form of schemes, including, for example: sample space (determines all the possible results of an event), experimental probability of an event (determines the probabilities of an event through experimentation, simulation and the use of relative frequency), and theoretical probability of an event (analyses all the possible results through the use of symmetry, numbers and geometrical measures in order to determine the probabilities of an event). Thus, it is possible to support the comprehension of theoretical probabilities through the enumeration of combinations.

To this end, an interpretive model of cognitive activities by DeBlois (2001b, 2003) is used to construe the meaning constructed by a given student. It promotes analysis of the evolution of the students' reasoning at school. It is a model that takes into consideration the student's representation of the situation (Brun, Conne, 1990), the procedures that she employs, and the expectations generated by the didactic contract, an implicit contract between the teacher and the students (Brousseau, 1997). The various modes of linkage between the components of the models can be brought out to trigger insights into ways of developing generalizable structuration of meaning. Such insights can be said to occur whenever students understand why they are able to solve a given problem, thus prompting them to adapt and mobilize their knowledge within a new context. At the same time, partial structuration of knowledge can also be developed. When they develop, students are able to solve some problems, yet remain captive of the problem's surface appearance, or of other components that prevent their knowledge from being adapted. We drew from this model in our exploratory research in order to analyse students' reasoning.

We also drew from works on critical thinking in our research because it seems to us an important point implicitly exposed in a validation situation as defined by Brousseau (1997). A student who has developed critical thinking explicitly uses the knowledge developed in one learning context in a new one. In addition, this student could identify some criteria to explain his choice. Lipman (2003) believes that critical thinking facilitates judgment[3] based on criterion and through being self-correcting and sensitive to the context. This theory could help us to analyse the criteria used in the class. The students, according to the context, could determine the preferred criteria. For example, is the usual requirement of taking order into consideration relevant to the proposed situation? Furthermore, in the process of adopting criterion, opportunities for argumentation are created. Contrary to mere explanations, argumentation justifies a point of view by taking into account the epistemic value of arguments — that is, their degree of certainty (Duval, 1991). Critical thinking can become or remain self-corrective depending on the self-assessment of

[3]Judgment is understood here as the development of opinions, inferences and conclusions, inclusive of problem-solving, decision-making and the examination of new concepts (Lipman, 2003).

one's mental acts and states as it occurs during the construction of enumeration procedures. It has three components (Guilbert, 1990): a set of thinking skills such as comparing or analyzing facts (Paul, Elder, 2001), affective dispositions such as being open-minded and respectful, and knowledge of the content, based on personal experiences and knowledge of the context (Bailin, Case, Coombs, Daniels, 1999). Reasoning is necessary when using critical thinking; however, it is possible to reason without using a critical thinking such as applying an addition to solve a problem without critically think about the problem.

3 Research Methodology

This study took place in a larger study, where the focus was on the gap between planning a lesson and enact the same lesson by a pre-service teacher in her field experience (DeBlois, Maleux, 2005). Thus, an exploratory research was performed in collaboration with a preservice teacher in primary education and her supervising teacher in a Grade 4 class (with 10-year-old students). The preservice teacher, the supervising teacher, the school's remedial teacher and the researcher met during the fall semester. These meetings took place before and after the lesson proposed by the preservice teacher. Discussion during the meetings focused on planning and carrying out a lesson with the aim of deepening reflection on teaching practices and student learning. For example, the preservice teacher planned to have a bowl with the names of pupils, had identify a problem to solve and planned to have students working in teams to realise exploration phase, and planned some counter-examples. After the lesson, the preservice teacher reflected on the lesson using the students' works and talks. Discussions were recorded and transcribed. The analysis of the type of adaptation (what was taught is different of what was originally planned by the preservice teacher) from the preservice teacher's perspective was analysed first. We found 4 types of adaptation: projective adaptations, withdrawal adaptations, normative adaptations and avoidance adaptations (L. DeBlois, Maheux, 2005). After this work on the preservice teacher's learning, we wanted to know more about what did her students learn in the lesson on probability, mainly because this lesson seemed challenging for her. Therefore, we analysed the Grade 4 students' activity focusing on probability taught by the preservice teacher and reported by her in the meeting following the lesson. We analysed the transcripts of the discussions, especially the one done after the lesson where the preservice teacher described, based on her memory, the entire process of what happened in the lesson. We analysed transcriptions and students' work using the DeBlois (2003) model.

The proposed lesson centred on probabilistic structures as viewed from a theoretical approach framed in terms of the relationship between the numbers of positive cases out of the number of possible cases of some event. The lesson also used more or less implicitly the concept of equiprobability. The day before the proposed lesson, the preservice teacher used a drawing in order to represent choosing students to be assigned to sport teams. She asked some questions about the probability of drawing a boy or a girl, which allowed students to so some enumeration of possible cases. The task proposed was culled from a textbook and was read aloud by the preservice teacher. It involved enumerating the results of a draw and applying a method serving to rapidly determine all the possible combinations: *"Put 5 blue tokens and five 5 red tokens in a bag. Draw 3 tokens. Which colour could be the tokens you drawn? How many outcomes are possible? Find a way which allows you to deter-*

mine quickly all possible combinations." In response to those questions, the teacher guide stated that: *"Here are the 4 possible outcomes: BBB, BBR, BRR, RRR."* So, students were looking about the combinations of coloured tokens, given the fact that even if the task involved 10 tokens, there are only just 2 colours. The task was about finding combinations, and the teacher's guide was clear that students should show those combinations, but the preservice teacher asked them to find a way to enumerate all possible outcomes without forgetting any of them. She wanted them to use a systematic method. So, she was looking for arrangements, but through the textbook, she asked them to find combinations without discussing the order at first[4].

To prepare for the draw of 3 tokens, the students were to place 5 red tokens and 5 blue tokens in a container. According to the supervising teacher and the preservice teacher, this task was a challenge for these students, who were not accustomed to representing probabilistic or combinatorics results. In the first phase, this activity was performed in teams of two, using tokens and a sheet. The students began by handling the tokens to perform draws or simulate them. Then, the teams of two were paired up to form teams of four in order to discuss the enumeration procedures used. Finally, the teams of four chose one enumeration procedure and made a poster presenting their results. No extra guidelines were provided to them. After the presentation of posters, the preservice teacher started a discussion by asking some questions about the probability of getting some combinations.

We offer an analysis of the procedures students represented on posters in relation to the work performed during the first phase, since the students' work in phase one contributed to the results of the second phase. We first identified the procedures used by the students on the posters and posed hypotheses regarding their representation of the situation. What was the task and what they were supposed to do? We analysed the mathematical ideas represented by the students. We analysed them according to their conceptual understanding based on Piaget and Inhelder's (1974) and Jones et al. (1999) theoretical frameworks about combinatorics or probability. We categorized first the student's works according to three categories of enumeration procedures defined by Piaget and Inhelder (1974). Our analysis focused on students' understanding of the concept. Then, we compared the posters with the work done by the students on the first part of the lesson. We used this analysis to discuss the enumeration procedures accordingly to the work of Jones et al. (1999). The discourse reported in the transcription helped us to discuss on critical thinking developed when students presented their poster.

4 The Challenge of Counting

The presented learning situation proposed the enumeration of the possible combinations of a draw. The learning intentions set by the preservice teacher's lesson were for students, "to process and categorize information, familiarize themselves with the terminology, and determine the probabilities." She asked that they would not forget some outcomes. This type of task was likely to shape the students' varying conceptions, identified by Fischbein and Schnarch (1997) as compound and simple events.

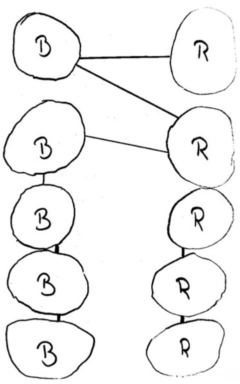
[4]Not discussing the order might lead to not consider the equiprobability to get those combinations. Because there is one way to get 3 tokens of the same colours and three ways to get 3 tokens of 2 different colours, it might mislead students that they have the same probability to get any of the 4 combinations. Due to the fact that the preservice teacher did not address this particular point, we decided to interpret essentially the combinatory activity.

We are speaking here of conceptions emerging out of a situation in which the students in the study confused the enumeration of combinations with the arrangement of repetitions. Numerous mathematical notions/concepts were drawn on: the terminology (probable, possible and certain); the notions of combination, arrangement and permutation; and enumeration processes.

As the students' posters suggested, the enumeration procedures used varied greatly. Owing to the theoretical approach that was experimented with, the students used three categories of enumeration procedures defined by Piaget and Inhelder (1974), that is: the successive constructions of trios; the juxtaposition of tokens; and the creation of an intersection of sets of tokens.

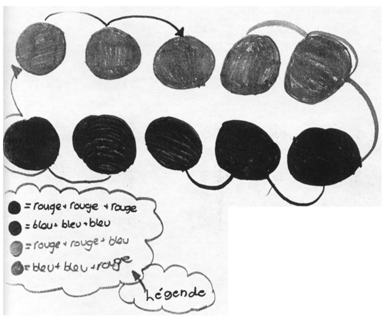
4.1 Successive construction of trios

Poster A



To represent the situation, the students responsible for poster A created a vertical diagram made up of lines and in which the combinations are identified by coloured lines: black (BRR), green (BBR), brown (BBB) and purple (RRR). They drew 5 blue tokens on the left and 5 red tokens on the right. This procedure is a successive construction of trios (Piaget, Inhelder, 1974). Although this answer is correct, it could lead to confusion or omissions in cases involving a larger number of tokens. When consulting the work performed by these students during the first portion of the exercise, we noted that one team of two students had used this procedure, while the other team had instead written four combinations horizontally and numbered them. We hypothesize that these students had sought to conform to the diagrams already presented in class, thereby complying with the "didactic contract". This search for conformity could hinder the development of an original enumeration procedure.

Poster B

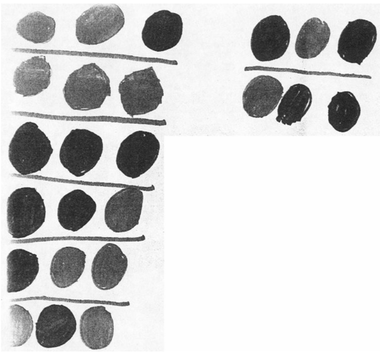


The representation of the situation found in poster B resembles poster A, because these students also drew on the successive construction of trios (Piaget, Inhelder, 1974). The diagram presented is horizontal: 5 red tokens above and 5 blue tokens below. The lines identifying the combinations are also of different colours: black (RRR), orange (RRB), purple (BBB) and green (BBR). These lines have arrows that indicate a direction. A legend indicating the colours and possible combinations is located in the lower left-hand corner of the poster. When we consulted the work performed by these students during the first portion of the exercise, we noted that one team had enumerated all the possible cases using the arrangement procedure. However, when these students got together with the other team, they decided to present combinations instead. We hypothesize that the procedure used by these four students drew on procedures already seen in other contexts and that the direction provided by the arrows indicated a search for order. The answer is correct but, as in poster A, the procedure used could lead to confusion or omissions in a case where a larger number of tokens were used.

In summary, posters A and B involve similar procedures. They represent diagrams in which coloured lines lead to the formation of combinations. While poster B presents a certain order through the direction of its arrows and by the inclusion of a legend indicating the combinations, the order of poster A is suggested by the arrangement of the tokens. In both cases, an interaction between the trios is suggested: each line leads to the following token, except when tokens are used a second time in each of the examples of poster.

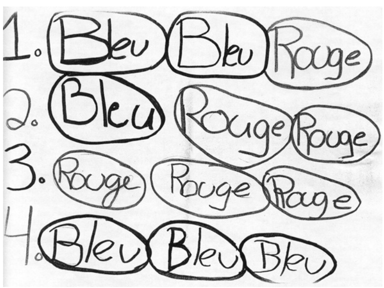
4.2 Juxtaposition of tokens

Poster C



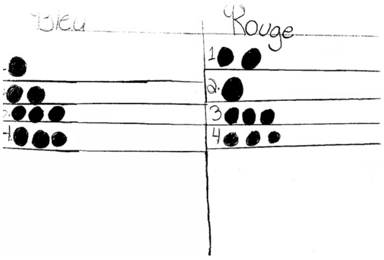
The students responsible for poster C represented the situation in a systematic manner, noting all the arrangements with possible repetitions. At first glance, the presentation displays no order. The first column presents a display of BBR, BBB, RRR, RRB, RBB, BRB. The second presents RBR and BRR. Perhaps they proceeded exclusively by draw. Upon analyzing the arrangements, we noted that the first four are in fact the expected combinations, while the four others are arrangements with possible repetitions. A coloured token juxtaposition procedure has been used along with an interaction of combinations (Piaget, Inhelder, 1974), but the result desired by the preservice teacher regarding a systematic method was not obtained. The presentation of the combinations presented leads to think of the use of symmetry. The work performed by these students during the first portion of their work involves a similar enumeration. In this instance, the combinations are presented in two columns of four, which suggest that after creating a combination, they determined the various possible arrangements of it. A distinction between combination and arrangement with repetitions could prompt the students to use a procedure that is more relevant to the context proposed.

Poster D



The representation of the situation by the students responsible for poster D reveals a systematic reading of the task. They presented the tokens on four numbered rows, each of which contained a different combination. This procedure of symmetrical juxtaposition of the coloured tokens enabled them to find all the expected results (Piaget, Inhelder, 1974). We hypothesize that this procedure was used without using a procedure presented in the textbook, e.g., a drawing. The work performed by these students in the first portion of their work showed an identical form of enumeration. This poster displays a procedure in which the risks of confusion or omissions are reduced, even when a larger number of tokens is used.

Poster E

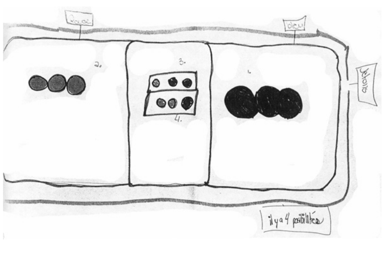


Poster E displays a peculiar way of representing the situation. There is a suggestion of some sort of order because the students started by dividing the sheet into two columns and titling them blue and red. However, these titles do not jibe with the contents of their respective columns. In the first column, they drew 1 blue token on the first line, 2 blue tokens on the second line, 3 red tokens on the third line and 3 blue tokens on the fourth line. In the second column, they drew 2 red tokens on the first line, 1 red token on the second line, 3 red tokens on the third line and 3 blue tokens on the fourth line. Each of the first two lines of the columns corresponds to a token juxtaposition procedure giving a single result: (BRR) and (BBR). The last two lines present a repetition of the results of each of the columns: (RRR) and (BBB). This does not correspond to the desired result because (BBB) and (RRR) are repeated. The students seemed to hesitate between juxtaposition and symmetry (Piaget, Inhelder, 1974). Our examination of the work performed by these students during the first portion of the exercise found that one team had created a similar table without repeating the 3 red tokens and 3 blue tokens in the columns, while the second team had created a two-way table.

In summary, the procedures used by the students responsible for posters C, D and E all drew on procedures based on the juxtaposition of tokens to construct a system that could be used to enumerate the possible cases. The token-juxtaposition enumeration procedure is based on an additive rather than multiplicative structure (Piaget, Inhelder, 1974). These procedures, which are necessary to the development of notions of probability, are based on a draw or a simulation of a draw in which tokens are moved around on the table. In addition, they highlight how an appreciation of the usefulness of enumeration procedures is dependent on the understanding that the students have of the task context. Thus, the students responsible for poster C wrote down arrangements with repetitions rather than combinations.

4.3 An intersection of sets of tokens

Poster F



The representation of the situation by the students responsible for poster F reveals a different interpretation of the task, because the students drew a diagram containing an intersection (Piaget, Inhelder, 1974). The right-most set contains 3 red tokens; the left-most, 3 blue tokens. The intersection of these sets contains two possible combinations: 1 red token with 2 blue tokens and 2 red tokens with 1 blue token. The combinations are numbered from 1 to 4: (1) the 3 blue tokens, (2) the 3 red tokens, (3 and 4) the combinations of tokens in the intersection. The students used a procedure that was inspired by other diagrams previously seen but in other contexts. Our examination of the work by these students in the first portion of the exercise indicated that both teams had drawn two-way tables: red and blue. However, one team had also drawn a diagram. This enumeration procedure displays the formation of a system constructed using a multiplicative method, with the intersection suggesting multiplicative structure (Piaget, Inhelder, 1974).

5 A Challenge for Teaching

5.1 What knowledge might contribute to the development of the students' enumeration strategies?

The students' posters bring into sharp relief the importance of analyzing enumeration procedures. They make it possible to hypothesize about the type of reasoning involved. We believe that the students who used procedures involving successive construction of combinations (posters A and B) and juxtaposition of tokens (posters C, D and E) should be ranked at the level of transition between subjective and naive quantitative reasoning (Level 2) of Jones et al. (1999). These students are able to enumerate the possible results but when they do so, they rely on additive structure notions such as: addition relationships, union of elements, groupings and complements of a set. According to Piaget and Inhelder (1974), these procedures are static invariants rather than successive interferences or transformations, which are dynamic. The group of students who used a procedure involving an intersection of sets of tokens (poster F) can be identified with informal quantitative reasoning (Level 3) of Jones et al. (1999). These children used a multiplicative structure, a kind needed to construct a fully operative system, thus creating the possibility of constructing the notion of relationship. Thus, associativity, like multiplicative structures, might be a useful knowledge when developing enumeration procedures, which themselves could help foster the development of probabilistic structures.

5.2 How can the procedures be developed in the classroom?

The challenge is to use an enumeration procedure relevant to the task-solving context. The distinction between combination and arrangement has an influence on the procedures used, since these notions are closely related and thus often become a source of confusion. The memorization of certain models could be a useful aid for students, just as it might well prove to be an impediment. When a memorized procedure is relied on, students are deprived of the conditions that would enable them to construct an original procedure of their own. The students must evaluate the procedures critically in order to use those that are not a source of confusion or oversight. Communicate their mathematical ideas is also a way to expose different procedures and think about them.

In the case of enumerating combinations, the students gained knowledge of the terminology on combination and the notions underlying this procedure and were thus able to go on and develop important aspects of probabilistic reasoning. This type of reasoning is the cornerstone of the representation of the situation, because, as Scholz (1991) emphasizes, it is possible to represent one and the same situation according to different approaches of probabilities. However, owing to the stages and characteristics inherent to cognitive development, a range of different processes or modes of thinking necessarily come into play. During those moments of institutionalization that occur in the classroom (Brousseau, 1997), an awareness was generated among students of the mathematical learning that they have achieved (Bruner, 1960, 1987; Piaget, 1974; Lucie DeBlois, 2001a).

5.3 An environment to develop critical thinking practices

The pooling of deliberations between small groups and the presentation of student solutions may foster the development of critical thinking regarding enumeration procedures. One likely manifestation of this process can be seen when a learner goes back over the instructions and the previously defined criteria and raises questions.

An example of such a manifestation occurred during a discussion with the preservice teacher following the presentation of her students' posters before the entire class. The preservice teacher described what happened: "The girls [a team of 2 girls] said 'we found more than 4 [combinations] and when we were paired with the other team, they told us that we had made a mistake' ... We [the preservice teacher, speaking for the whole class] went back over what would have been important to specify from the start. Do we take order into consideration or not? We thought about what changes were entailed by our response. Then we did the exercises over again but did not specify whether or not we were taking order into consideration. But several hands were raised by students asking 'do we take order into consideration or not'?". Some students were pushed to think about the importance of order during and after this learning activity. For them, order opened the way to permutations and to enumeration of arrangements with repetition. When instruction provides the opportunity for students to think critically with respect to the task-solving context, the result is to trigger a review of the circumstances and limits of an event with a view to working out a broader view of the proposed problem situation such that order can be used meaningfully. From this perspective, exposing students to a variety of contexts is very likely to be worthwhile in terms of stimulating the emergence of comparisons between procedures.

Furthermore it is important to emphasize the influence of affective dispositions, which, alongside knowledge, are constitutive of critical thinking (Guilbert, 1990). Such affective dispositions appear during the second phase of the activity. Following a two-fold analysis of the pooling of the posters of the $2+2$ teams, on the one hand, and of the selection of a procedure for presentation to the class, on the other, could prove a basis for explaining why many students preferred to repeat a previously produced poster than to discuss a new procedure. In that regard, consider posters A, B and E. Our analysis has shown that the procedure used in poster A and B is rather rudimentary and that the procedure used in poster D is more sophisticated. Two students in team A previously used a token juxtaposition procedure rather than a successive construction of trios. However, they decided to present a successive construction of tokens. A search for conformity with known models may have acted as a criterion in the selection of a form of presentation. It is also possible that poster A gives the impression of being sophisticated because there is a diagram.

6 Conclusion

In this study, students represented the situation in order to use a relevant strategy according to the context. They had to find a way to represent all possible combinations of a drawing using 3 tokens of two different colors. Five teams of four students were able to do it using additives structures: addition relationships, union of elements, groupings and complements of a set. One team of four students used multiplicative structures: intersection of sets of tokens. These results suggest that the reasoning involved in enumeration procedures is in fact more complex than it appears at first glance and might involves associativity, as well as multiplicative structures. More complex reasoning was employed in selecting these procedures, which were in fact enumeration strategies. Teaching these procedures themselves may not lead students to use them as a strategy to solve problems according to the context of the situation. Providing situations in which it could be relevant to use them as well as a structure to discuss the choice of strategies could be a way to support students to develop increasingly sophisticated reasoning, a new way of thinking teaching probability.

This type of situation involves the use of critical thinking. In fact, discussions between students when they worked in teams of 2 and later in teams of 4 and plenary discussions allowed them to share their reasoning about strategies employed, and thus, develop their critical thinking by the use of a variety of arguments and by the creation of the criteria to accept or refuse these arguments. It is when students gave themselves a criterion about the order that awareness about probabilistic reasoning occurred. Then, they might determine the probability of an event depending on the order. Critical thinking contributes to probabilistic reasoning and it must be emphasized in classrooms. It seems that critical thinking must be integrated in habitual activities. But how should critical thinking be taught? And how can critical thinking contribute to develop more abstract reasoning about probability? These points need to be highlighted in further work on critical thinking and the didactic of probability.

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