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Co bychom měli požadovat od toho, kdo učí matematiku na 1. stupni školy?

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Abstrakt

Článek je rozšířenou verzí plenární přednášky na konferenci SEMT 11 (Vinner, 2011). Jsou zde rozpracovány některé otázky, kterým se autor ve své přednášce z časových důvodů nemohl věnovat. Autor doporučuje, aby kromě znalostí matematiky potřebných pro výuku matematiky na 1. stupni základní školy byla dostatečná pozornost věnována také dalším aspektům práce učitele. Doporučuje, aby důvody pro zařazení matematiky jako povinného předmětu pro celou populaci, stejně jako základní cíle vzdělávání byly diskutovány s učiteli. Doporučuje také, aby z přípravy učitelů byla vyřazena témata, která jsou vně ZPD (Zone of Proximal Development – zóny nejbližšího vývoje).

Klíčová slova: příprava učitelů matematiky pro 1. stupeň základní školy, profil učitele elementární matematiky, složky znalostí učitele elementární matematiky.

What Should We Expect from Somebody Who Teaches Mathematics in Elementary Schools?

Abstract

This paper is an extension of my plenary talk in SEMT 11 (Vinner, 2011) in which I promised to elaborate on some issues that I could not discuss at the talk because of time restrictions. The paper recommends that, in addition to the mathematical knowledge needed for teaching in elementary schools, attention should be given to other aspects of the teacher's work. It is suggested that the rationale of teaching mathematics as a compulsory subject to the entire population should be discussed with the teachers as well as the ultimate goals of education. The paper recommends also avoiding in teacher training topics which are beyond the zone of proximal development (ZPD) of the teachers.

Key words: elementary teachers education, an elementary teacher's profile, aspects of an elementary teacher's knowledge.

1 Prologue

The theme of SEMT 11 was the mathematical knowledge needed for teaching in elementary schools. The title of my paper indicates that I have chosen to discuss the teaching mathematics in elementary schools in a broader context, emphasizing additional aspects of teaching mathematics. I believe that, all over the world, teacher training at this stage has adopted the approach that content knowledge and pedagogical content knowledge should be taught to those who prepare themselves to teach at the elementary level, as well as to teachers who come to us for further studies after serving for a while in the schools. In addition to the content knowledge and the pedagogical content knowledge, there is a consensus that elementary teachers should know something about children's mathematical thinking. They should be aware of the causes for typical mistakes and should be able to understand children's original ideas about doing mathematics, whether these ideas are correct or incorrect. This is in fact the message of Ball, Hill and Bass (2005). It seems that the majority of people who are involved with mathematics teacher training will agree with the above, but when it comes to details, the number of opinions is almost equal to the number of people who are involved in the domain. There are quite obvious reasons for that:

1.1 As to the content knowledge

The content knowledge depends on the mathematics curriculum which is supposed to be covered at the elementary level. Different countries may have different curricula. Even in the same country the curriculum keeps changing over the years. It turns out that mathematics curriculum people have not reached an agreement about questions like: should we or should we not teach combinatorics or probability at the elementary level, or at what grade are we supposed to teach fractions or negative numbers? During the last fifty years the elementary mathematical curriculum has been overloaded with some mathematical topics which are beyond the zone of proximal development (Vygotsky, 1986) of the elementary pupils. In addition to that, the dominant approach of the mathematical education community is that mathematical rules and procedures should be explained to the pupils. However, the real mathematical explanations are, sometimes, beyond the pupils' zone of proximal development. Thus, some mathematics educators came up with alternative explanations which, supposedly, would be understood by the pupils. Unfortunately, some of these explanations are quite ridiculous. Their contribution to mathematics education at this level is rather negative. In this case, should we or should we not eliminate these mathematical topics from the curriculum? Should we or should we not give up the principle that every mathematical rule should be explained? At this point we can see that content knowledge decisions sometimes depend on pedagogical content knowledge available at a given moment. As claimed above, the number of opinions about these issues is almost equal to the number of people who are involved in the domain.

1.2 As to the pedagogical content knowledge

Different mathematics educators with different backgrounds or culture often have different opinions about the clarity and the efficacy of certain "real life models" which are supposed to explain why some mathematical operations are as they are.

The following INTERNET excerpt (American_educator/fall99) about multiplying negative numbers can clearly support my claim:

Question: *Are there any good websites or other resources to help explain negative times negative numbers?*

Reactions:
(I). Ds *is going to through The Key to Algebra, Book 1, and it uses a football field explanation... He does not know much about football and it is confusing.*
(II). Dd *said her math book used football as well (Scott Foresman). She knows very little about football and feels his pain. I think she just memorizes.*

1.3 As to children mathematical thinking

There is a huge literature about it (many thousands of pages), how much of it and what exactly are we supposed to present to our teachers and prospective teachers?

I am going to address shortly these issues but my main emphasis will be on other four questions which, in my opinion, are quite important (if not crucial) for the elementary teachers' work:

1.4 Why do we teach mathematics?

1.5 What is mathematics?

1.6 What is mathematics education?

1.7 In what ways does the teaching of mathematics serve the ultimate goal of education?

An additional issue to all the above but an essential one is the question:

1.8 To what extent the elementary teachers have the necessary background to study what we expect them to know so that they will be able to implement the tasks that the educational system presents to them?

I would like to discuss first the last question since the answers to it determine the answers to the previous questions.

2 Typical profiles of elementary teachers

Elementary teachers have many profiles. They have many profiles even when speaking about one country. Since this is an international conference it seems that I were supposed to speak about several countries. In order to do it, I were supposed to rely on an early survey in these countries, and to present different profiles with their different distributions. Since there is no such a survey, I have chosen to present to you some anecdotal profiles of elementary teachers in my country and in the USA which were picked up from my own experience and from the mathematics education literature. Being concerned about the international relevance of my talk I would like to suggest the following justification for presenting such a local picture:

a) There might be some similarities between the above two countries and other countries, and therefore what I am saying might be relevant to other countries as well.
b) The situation in other countries might be different than the situation in the above two countries. In this case, it is interesting to learn about other countries like tourists who visit other countries in order to see different landscapes.

My anecdotal impression of talking to elementary school teachers (especially in grades 1–3) in my country and in the USA is that they decided to become elementary school teachers because they like the human interaction with little children. Being involved with the children's intellectual and emotional development gives them a lot of satisfaction. Usually, they do not score very high on the college entrance examinations. For me, mentioning it is quite problematic. It is problematic because I cannot ignore the fact that this claim is expressed by some people with arrogance. It is also problematic because it suggests forming hierarchies of people by comparing their scores on college entrance exams. I myself do not believe in measuring people. The title of a canonical book on this issue is *The Mismeasure of Man* (Gould, 1981). To the educational community I would like to suggest even a stronger title: *The Immeasurable Man*. Modern psychologists speak about many kinds of intelligence: emotional intelligence, social intelligence, and more (Gardner, 1993; Goleman, 1995). These kinds of intelligence are not less important to the success of teachers than their cognitive intelligence. Yet, there do not exist tests to measure these kinds of intelligence. It is not that I do not understand the need to evaluate the ability to study a certain domain when someone wants to study it at a college. However, since mathematics is an important component in college entrance exams, the fact that prospective teachers do not score high on college entrance exams may predict that they will have some difficulties studying certain mathematical topics. Supervision and in-service courses reveal quite often the mathematical weakness of some elementary teachers. For example, I am told by colleagues who teach remedial courses to these teachers that a significant number of them have difficulties in solving word problems like the following:

1. *David holds 5/8 of the shares of a certain factory. He gives his son Daniel 2/3 of his shares. What part of the factory shares is owned by Daniel after this transaction?*

2. *Barbecuing meat causes it to lose 1/5 of its weight. What was the original weight of a piece of meat if after barbecuing, its weight was 300 grams?*

In a study (Guberman, 2007), based on an adaptation of Van Hiele four geometrical levels to arithmetic, it was found that 63 % of the student teachers were below the third level in the beginning of the college arithmetic course. Only 4 % of these students improved their location in the four level hierarchy at the end of the arithmetic course. Similar results were obtained in a study by Pandiscio and Knight (2011) which examined *the van Hiele level of geometric understanding of pre-service mathematics teachers, both before and after taking the geometry course required by their teacher preparation program. Results indicate that prior to the course, pre-service teachers do not possess a level of understanding at or above that expected of their target students… the magnitude of the gains (obtained by the end of the course) was not enough to raise the sample population's van Hiele level to that expected of their future K-12 students.* There are more studies in which similar results were obtained but I am not mentioning them here because of space restrictions. I would like to conclude this section about elementary teacher profiles with a quotation from Ball, Hill and Bass paper (2005): *many U.S. teachers lack sound mathematical understanding and skills... Mathematical knowledge of most adult Americans is often as weak, and often weaker.*

3 Some recommendations

From what I have said till now it follows that it is impossible to suggest a uniform list of mathematical topics that prospective teachers should study while preparing themselves to become teachers at the elementary level. However, I would like to suggest three pedagogical principles which should lead curriculum designers and teacher trainers at the colleges of education.

i) Ausubel's leading principle: "If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. vi). In other words, when instruction is designed, the starting point of the student should determine it. This implies that if the mathematical background of the student is poor we should first improve it and only later, move on to more advanced topics.

ii) The zone of proximal development principle (Vygotsky, 1986). Adapting the zone of proximal development principle to our situation implies that we should not try to teach our students topics which are beyond their intellectual ability. It is worthwhile mentioning that the notion of the zone of proximal development is quite vague. Namely, even if we know "what the learner already knows" we might have difficulties predicting whether the learner is able to cope with a given topic which presumably belongs to the learner's ZPD. For instance, assuming the learner is familiar with the concept of rational numbers. Will he or she be able to learn meaningfully the concept of irrational numbers?

iii) The suitable pace of teaching. There is a general tendency to overload syllabi and then, because of the unwritten obligation to cover them, the pace of teaching is too fast for the decisive majority of the learners. However, ignoring the above principles leads to meaningless learning. Meaningless learning expresses itself very often in what I call *pseudo conceptual and pseudo analytical behaviors* (Vinner, 1997). I will elaborate on it later on. At this stage I would like to claim that teaching something which we do not really understand is disastrous. However, this is the case with many elementary teachers.

Coming back to issues 1.1–1.3 in the prologue, my recommendations are the following:

As to mathematical content knowledge (1.1), the above three principles can help us to determine a list of mathematical topics which can be presented to pre-service and in-service elementary teachers in different social and cultural settings. As explained earlier, these principles cannot lead to a uniform universal curriculum. Giving up a uniform universal curriculum is unacceptable by the majority of influential people involved in national systems of education. Unfortunately, *because of the comparative international surveys in science and mathematics, education has become an international competition.* Educators and educational policy makers can argue about the advantages and the disadvantages of this fact. However, if as a result of this competition a uniform mathematical curriculum for elementary pupils will be suggested, which will imply also a uniform mathematical curriculum for elementary teachers, it will not help mathematical education. We cannot overcome individual differences by uniform curriculum. On the contrary, individual differences should be taken care of by differential curricula.

As to pedagogical content knowledge (1.2), I recommend that concrete models and representations should be used only if they are simple and clear. This is true for elementary teachers as well as for elementary pupils.

And as to *children mathematical thinking* (1.3), since there is no canonical list for this topic I suggest that we should prefer clear, simple and straight forward texts to more sophisticated and complicated studies. For obvious reasons I am not mentioning any specific texts.

4 Why do we teach mathematics?

I consider this question as a metacognitive one. Namely, elementary teachers who teach mathematics do not raise it and therefore, they do not have to answer it. They teach mathematics because it is part of the curriculum which they are supposed to teach. If the question is raised by an external agent there are ready made answers to the question. Usually, curriculum designers present a rationale for teaching the curriculum which they recommend. The most beautiful rhetoric for teaching mathematics which I know is the NCTM (2000) rhetoric.

We live in a mathematical world, whenever we decide on a purchase, choose insurance or health plan, or use a spreadsheet, we rely on mathematical understanding... The level of mathematical thinking and problem solving needed in the workplace has increased dramatically... Mathematical competence opens doors to productive future. A lack of mathematical competence closes those doors.

This is not the place to elaborate in length how misleading these claims are. In short I will say only the following: No doubt mathematical knowledge is crucial to produce and maintain the most important aspects of our present life. This does not imply that the majority of people should know mathematics. Farming is also crucial to at least one aspect of our life — the food aspect, and yet, in developed countries, about 1 % or 2 % of the population can supply the needs of the entire population. In addition to this argument, if you are not convinced, I recommend to you to look around and to examine the mathematical knowledge of some high rank professionals that you know — medical doctors, lawyers, business administrators, and many others, not to mention politicians and mass communication people. Recently, an attempt to refute the above claim that *the level of mathematical thinking and problem solving needed in the workplace has increased dramatically* came from an unexpected source, a research mathematician, Underwood Dudley (2010), who sampled randomly from the *yellow pages* 8 categories of work places and found no evidence that algebra is required there, *"even for training or license"*. Another claim in the rhetoric which is supposed to justify the teaching of mathematics is the claim that mathematics is needed for everyday life. However, whenever I ask mathematics teachers who claim it for specific examples the only examples they come up with are calculating tips in restaurants, calculating change (this concerns mainly taxi drivers) and cooking (calculating the amount of ingredients for n people, when the amount of ingredients for m people is given in the cook book, $m \neq n$). There might be other convincing arguments to study mathematics. Underwood Dudley claims that people should study mathematics in order to train their mind. However, there is no experimental evidence which supports the claim that, in non-mathematical domains, people who studied mathematics are better problem solvers than people who did not study mathematics. Another possible reason for studying mathematics is the application of certain mathematical chapters in sciences (physics, chemistry, biology, etc.). The question here is to what percentage of the population this claim is relevant and whether there are no other ways to reach this percentage rather than imposing mathematics on the entire population. Thus, if the above claims about the need to study mathematics are misleading, why do we teach mathematics and why do our students study mathematics in spite of all? You might suggest that the students believe in these claims although these claims are misleading. I suggest that the students have very good reasons to study mathematics. It is not the necessity of mathematics in their future professional life or in their everyday life. It is because of the selection role mathematics has in all stages of our educational system. Mathematical achievements are required if you want to study in a prestigious place (whether this is a junior high school, a senior high school or university). A prestigious school increases your chance to get a good job. Confrey (1995) formulated it quite clearly: *In the vast majority of countries around the world, mathematics acts as a draconian filter to the pursuit of further technical and quantitative studies...*

Eventually, we have a convincing argument to study mathematics. Should we tell it to pre-service and in-service teachers? I believe we should. It is important for a soldier to know the real purpose of a battle in which he or she takes part. He or she should be convinced that there are good reasons for risking their life. Intentionally false rhetoric should be morally unaccepted. I am not claiming that there was a conspiracy to form a false rhetoric about the need to study mathematics. On the contrary, I think that the people who invented this rhetoric really believed in it *bona fide*. However, beliefs should be re-examined from time to time. The main thing is that teachers will have worthy goals for their endeavor. Is preparing students for crucial examination a worthy goal? I believe it is. Both students and teachers are victims of the same educational reality and as far as we can see, the chance to change this reality is very small. For a great part of the younger population, to continue their formal education (generally, not in a domain that requires mathematics) is an important goal. Pupils are expected to progress from the elementary level through the junior high level to the high school level and then to college and university. At crucial points of this journey, there are guards who examine them on mathematics. If the pupils pass the exams the guards let them move on. It is a worthy goal to help pupils complete this journey. Of course, there is much more into mathematics. There are intellectual values and educational values. Usually, because of the common way mathematics is taught, pupils are not exposed to it. I will come back to this point later on.

5 What is Mathematics?

This is again a meta-cognitive question. Generally speaking, people do not seek definitions for the notions they use. The meaning of the decisive majority of concepts in everyday thought is determined by means of examples and not by means of definitions. This was explained extensively in Vinner (2011). Even some mathematicians, when being asked what mathematics is, prefer to give examples. Among them I can mention Courant and Robins (1948). Their book is full of mathematical examples. They probably believed that people, who were not mathematics majors but who were eager to know what mathematics is, would be able to understand the mathematical chapters which were presented in the book. Another book which deals with this question is Hersh's book (1998). This is a philosophical book which presents mathematics as a human endeavor. However, there is no attempt in this book to define mathematics as a generalization of specific examples. I believe that (in contrast to other concepts like poetry, art, etc.) it is possible to suggest a definition of mathematics which is a generalization of specific examples. It is true that in order to understand this definition one should have at least a Bachelor degree in mathematics. For this reason and also because of the technical nature of the definition I am presenting it in appendix II. However, in case we want to avoid technical difficulties, the best place to look for simple and general definitions is the dictionary. The Webster's Ninth New Collegiate Dictionary suggests that Mathematics is *the science of numbers and their operations, interrelations... and of space configurations and their structure...* The mathematics which is characterized by this definition includes only arithmetic and geometry. It is not clear whether school algebra is included. Elementary school teachers are supposed to know these branches of mathematics. If you ask them what mathematics is, will they be able to give a similar definition to the one given by the dictionary? My hypothesis (based on the claims in Vinner (2011)) was that the elementary mathematics teachers will not be able to give an answer which is close enough to the definition given in the dictionary. Their answers, according to this hypothesis, should somehow reflect the examples of mathematics they had in their past and present experience. And indeed, out of 120 teachers to whom the questionnaire which I designed (see appendix I) was distributed, no one gave an answer resembling the dictionary definition. Thus, we can assume that their idea of mathematics is determined by the mathematical experience they had in school and by the mathematics they teach. Therefore, if you generalize the past and present mathematical experience of these teachers it is quite reasonable to assume that their view about mathematics is that *it is a collection of procedures to be used in order to solve some typical questions given in some crucial exams (final course exams, psychometric exams, SAT etc.)* This is how it was described in the questionnaire. And by the way, trying to reconstruct my own implicit views about mathematics which were based on my experience as a high school student studying an extensive mathematics curriculum — I believe it was quite similar to the above. When you present this view to mathematics teachers by means of the above mentioned questionnaire they notice the negative connotation of this statement and they try to reject it. However, the arguments they suggest in order to reject it are usually the following: *(I) Mathematics is not only for exams, it is also for real life situations. (II) Mathematics teaches you to think.* When being asked to specify, in most of the cases, they use their right to remain silent. My question is whether we should tell the elementary teachers *what really mathematics is?* But before answering this question we should find out whether these teachers have the required mathematical and intellectual background to understand the answer.

6 What is mathematics education?

The following section is, perhaps, more suitable for elementary teachers who are studying or planning to study for a master degree in mathematics education. The notion of mathematics education is both ambiguous and vague. For some people mathematics education is the production of text books, learning materials, mathematics curriculum design, enrichment of the mathematics curriculum, developing software by means of which certain mathematical topics can be taught, recommending teaching methods and so on. Erich Wittmann, in his plenary talk at the International Symposium on Elementary Mathematics Teaching, SEMT 11 (August 21–26, 2011), repeated a 16 year old claim of his that *Mathematics Education is a design science* (Wittmann, 1995). Since there are some criteria which certain activities should meet in order to be considered as science, and I am not sure whether the activities at which Wittmann pointed fulfill these criteria, I would like to use the word "design" in a different way. Note that this word is used intensively in many practical domains such as: hair design, fashion design, industrial design, interior design and many more. I prefer to speak about the design of instructional materials for the learning and teaching mathematics. On the other hand, for other people, mathematics education is a research discipline which investigates the learning and teaching of mathematics. It relates among other things to mathematical concept formation, to thought processes which occur in students while they solve mathematical problems or when they prove mathematical claims. It also investigates the dynamics of mathematics classes and teacher–student interactions. For some people it means both. Namely, they try to design all kinds of innovations for mathematics classrooms and they also investigate the impact of these innovations on the students. I claimed that "mathematics education" is also a vague notion because there is no consensus about what topics should be included in the domain and what topics should be excluded from it. For instance, should the question where mathematics came from be discussed within the domain of mathematics education or not? I do not want to discuss this question here. I have mentioned it only for the sake of bringing it to the reader's attention

Quite often, when I am asked about mathematics education by people who are outsiders to this domain, I am trying to compare mathematics education research to medical research. We study diseases in order to find for them a cure. If the analogy to medical research is extended it is worthwhile to note in it three stages. The symptoms, the diagnosis and the cure. The symptoms in our domain are that the majority of people who have to study mathematics hate it. After going through the compulsory mathematics courses, they consider themselves as mathematically ignorant and some of them are even proud about it. Although the domain of mathematics education is offering various cures to the disease, my opinion is that we are still at the diagnosis stage. However, it will be wrong to ignore some wonderful text books and software in which there is a real promise to improve mathematics education and to enhance meaningful learning. Unfortunately, these materials have not become the common materials for the learning of mathematics. They are still considered as experimental materials used in relatively small populations.

7 In what ways does the teaching of mathematics serve the ultimate goal of education?

Unfortunately, thousands of pages in educational philosophy have been written about the ultimate goal of education. It has also been the theme of hundreds of educational conferences. I say "unfortunately" because the too many trees prevent us to see the forest. Therefore, I would like to suggest a simple answer to this question. The ultimate goal of education is an educated person. This is, of course, circular. In order to avoid it, I would say that an educated person is *a thoughtful person.* "Thoughtful" in English is ambiguous. The above Merriam-Webster dictionary suggests the following: *a) characterized by careful reasoned thinking... b) given to heedful anticipation of the needs and wants of others.* In other words, "thoughtful" also means "considerate". This can be tied to, what is called in moral thinking, the *golden rule.* There are plenty of versions for this rule which come from various cultures and religions. One Jewish version of it is: *What you hate — do not do to other people.* In order to follow this rule you should first apply your *careful reasoned thinking.* Namely, you should carefully analyze everyday situations and determine whether acting in a certain way in these situations will be unpleasant or even harmful to other people. Then you should control yourself and abstain from acting in such a way.

Earlier I mentioned that mathematics education, the way it is taught in the majority of schools, focuses mainly on mathematical procedures by means of which typical questions in typical exams can be solved. Mathematical procedures have negligible importance in everyday life and in the majority of work places. However, procedures in general, play crucial role in everyday life and in all work places. By "procedure" I mean a sequence of actions which should be carried out one after the other. Crossing streets, driving, shopping, turning on dishwashers, dryers, DVD players (etc., etc.) are all associated with procedures. This is just an accidental choice out of an infinite list of procedures. Thus, respecting procedures as well as carrying them out precisely and carefully can be recommended as an educational value. Note that not following certain procedures is against the law. One example out of infinitely many: Crossing an intersection in a red light while driving a car. Not following other procedures can result in an economical damage. Again, one example out of infinitely many: Not turning off all the lights and electrical instruments when leaving home. Note also that many procedures in everyday life were formed in order to serve the *golden rule.* For instance: procedures related to behavior on lines, procedures related to pedestrians and drivers and procedures related to littering and recycling. Teachers, while teaching mathematical procedures can point to the pupils at procedures in everyday life and speak about the importance of following these procedures precisely and carefully, the same way as required in mathematics. By doing this, teachers add educational value discussions to their traditional role which is to cover the syllabus. Within the traditional role the teacher is a tool of the syllabus. By adding educational discussions to the syllabus, the syllabus becomes an educational tool.

There are many other contexts in the mathematical curriculum where educational value discussions can be integrated. For instance, very often, when dealing with mathematical problem solving we speak about the need to apply analytical thinking and control to the problem solving process (see for instance Schoenfeld, 1985). We certainly speak about it in mathematics education conferences. Some of us may speak about it even in their mathematics class, in case we teach mathematics. It is considered as a meta-cognitive activity or a reflective activity. In teacher training we encourage our students to include it in their future teaching. I assume that very few of us and very few mathematics teachers, when they speak about it in their mathematics classes, point at the fact that they are also relevant to our everyday life and not only to mathematical thinking. They are also educational values which are directly related to the above mentioned golden rule. While speaking about analytical thinking and control we can tell our student teachers that analytical thinking and control are important factors of rational thinking and thus introduce to them the notion of rational thinking.

Rational thinking is strongly related to scientific thinking; however, it is broader than it. Mathematics is closely related to the development of science, and thus we have another justification to claim that mathematics is part of rational thinking. Rational thinking is the kind of thinking which is needed to maintain our society. By "our society" I mean the liberal democratic society with its values, its various institutions, its science, art, technology and medicine. A sophisticated characterization of rational thinking is beyond the scope of this paper. However, in everyday discourse, people use this notion and it is quite clear to them what it means. "Behave rationally," they recommend quite often to each other. The Merriam-Webster dictionary suggests that to be rational is to be *reasonable. Rationality is the quality or state of being agreeable to reason. Rationality is applied to opinions, beliefs and practices.* About being reasonable, the dictionary adds that reasonable is *not extreme or excessive* and it is *moderate and fair*. Wikipedia, the free encyclopedia of the Internet, claims that "rational", *in a number of kinds of speech, may also denote a hodge-podge of generally positive attributes, including: reasonable, not foolish, sane and good*. Note that both the Merriam-Webster and the Wikipedia agree that rationality, in ordinary language, has also a moral aspect (moderate, fair and good). Thus, the notion of rationality that I am suggesting to include as one of the educational goals of teaching mathematics includes also the moral aspect of rationality. Thus, again, we are involved in integrating educational values aspects in the teaching of mathematics. As an additional characterization of rationality, I would also like to emphasize that to *be rational implies taking into account science, medicine and technology*. People behave rationally if they take into account all the scientific information which is relevant to their decision making. Thus, rationality is a relative notion. A rational behavior in Newton's era is not necessarily rational in our era, since science has been changed dramatically. There is a broader discussion of these issues in Vinner (2007, 2008).

All the above were *general frameworks* within which value discussions can be integrated. In the section below I will describe some word problems which call for a mathematical treatment and can be extended to a value discussion. The first word problem is taken from Peled & Balacheff (2011). It is the following:

Andrea and Bill bought a $5 lottery ticket together. Andrea paid $3 and Bill paid $2. They won $40. How will they split it?

The question was presented to Ron, a sixth grader by his mother at home, while spending time together on various word problems.

Ron suggested three solutions:

1. *Split the win evenly.*
2. *Split it so that the difference is close to the difference between the "investments": Andrea gets $21 and Bill gets $19.*
3. *Split it proportionally: Use the investment ratio* 3 : 2. *Andrea gets $24 and Bill gets $16*

Finally, he commented: I think the first solution is the "most fair", but the third solution is the "most right" because it uses ratio. What Ron might be saying is that in real life, if he were in a situation like that, he would have used an even split. Since he knows that the teacher expects him to solve this problem using proportion, this is what he has to accept as what is considered a good mathematics class solution.

I consider this excerpt as a wonderful illustration how moral thinking can be involved in the mathematical thinking of an 11 year old child. Undoubtedly, in this child's value scale, equality has a clear priority. Andrea and Bill are friends. Because of various reasons Bill paid less than Andrea for the ticket. However, because of the friendship, they are supposed to split the win evenly. If however, the difference between the investments should be taken into account when splitting the win, then Andrea is supposed to get a little bit more then Bill. Thus, the amounts $21 and $19 seem quite fair to Ron. The seemingly correct mathematical solution is not considered by Ron to be a fair split. Mathematically, he is smart enough to produce this solution but he does not consider it fair. Note that in our adult community, splitting the win according to the ratio between the investments is considered a fair split. What are we supposed to tell the child who disagrees with us? I leave this as an open question to the reader and I assume that various answers can be given here, expressing the moral value scale of different readers.

The next problem is quite similar to the previous one but more complicated. It is taken from a Jewish wisdom book (the title of which is "The Seeker" and it was written in 1263 by Rabbi Shem Tov Ibn Falkira). It was brought to my attention by Yaari (2005). Here is Ibn Falkira's story updated by myself to our 2011 reality:

Two friends, Reuven and Shimon, after having a walk at the countryside, stopped at a bench in order to have their lunch. For lunch they brought 5 rolls. Reuven brought 2 of them and Shimon brought 3. A stranger passed by and asked them whether he can join their meal. They invited him generously and split the 5 rolls evenly between the 3 of them. After finishing their lunch the stranger took $5 out of his pocket, gave them to the friends, thanked them, and while walking away said: "Split what I gave you fairly and cleverly."

"I'll take my share now," said Reuven asking Shimon to give him his $2.5. "Wait a minute," said Shimon, I brought to our lunch 3 rolls and you brought only 2, hence your share is only $2 and my share is $3.

Being unable to solve the conflict they agreed to take their case to the court. The judge listened to their story, thought a little bit and announced his verdict: Reuven will get $1 and Shimon will get $4.

Please, explain.

This story has an additional dimension which was not present in the previous one. Namely, how much of Reuven's rolls and how much of Shimon's rolls was given to the stranger? Since the rolls were divided evenly so that each diner got 1 roll and 2 thirds of it, it is clear that the guest got 1/3 of a roll from Reuven and 4/3 of a roll from Shimon. Hence, the $5 should be split according to the ratio between 4/3 and 1/3.

After giving this explanation the moral issue can be raised for a class discussion as in the previous case. The fact that the verdict was $4 for Shimon and $1 for Reuven reflects the author's view about the "fair and clever" split. It is quite similar to the fairness principle of the textbook, as conceived by the 11 year old child in the previous story. Here there are 3 moral approaches to the above situation, and from the moral point of view it is impossible, in my opinion, to determine which one is better. Thus, practically, in order to avoid such conflicts, the principle of splitting the profit should be agreed upon by an early agreement, or by laws of the society to which the arguing people belong.

Another example of a specific word problem that can be a trigger for an educational value discussion can be found in Taplin (2002, p. 79).

The bottom line of this section brings me back to sections 2 and 3 in the beginning of my talk (the typical profile of the elementary teachers and to the recommendations related to it). While there are serious doubts about the feasibility of presenting to elementary teachers and prospective elementary teachers some of the topics mentioned above, there should not be any doubt about asking them to work toward the ultimate goal of education, namely, asking them to be educators. Unfortunately, in some societies, teachers (and especially elementary teachers) are blamed for not having satisfactory knowledge in science or mathematics. However, if we focus on educational aspects of the teacher's work then the above accusation becomes minor. Moreover, a counter accusation should be raised against parents who do not care so much about the education of their children but care a lot about their mathematical achievements; not because these achievements are really important for the future life of the children (as adults, 90 % of them belong to the population which does not use mathematics, hates mathematics and does not know mathematics), but because mathematical achievements are required for further academic or technical studies. And as to the elementary teachers — imposing on them mathematical demands which are beyond their mathematical abilities or, if you wish, beyond their ZPD, will bear negative results as explained in the next section.

8 The pseudo-analytical and pseudo-conceptual behaviors as a reaction to exaggerated intellectual demands

In an early work of mine (Vinner, 1997) I explained in length what I mean by these two notions (pseudo- analytic and pseudo-conceptual behaviors). Here I will say very shortly that these behaviors are produced by people who try to show that they know a certain topic but as a matter of fact they do not know it (in some cases, people really believe they know while they do not know. In other cases, they know that they do not know but they pretend to know). Here are three anecdotes:

(I) The supervision on mathematics education in my country has decided that all elementary teachers should know certain chapters in probability. Thus, teachers who did not have this knowledge were invited to participate in a compulsory course in which some elementary concepts in probability were introduced to them. At the end of the course, among other questions, they were asked to solve the following question: *There are 16 cards in a box. Each card is in an envelope. All the numbers between 1 and 16 (included 1 and 16) appear on the cards (one number per card). Describe an event the probability of which is 1/2.* Non-negligent number of the teachers suggested that a right answer to this question was to pull out of the box the card which has 8 on it. When being asked to justify their answer they said: Because 8/16 is 1/2. Superficially, it looks as a convincing analytical argument. I classify it as pseudo analytical behavior.

(II) The second anecdote is about an elementary teacher who taught her pupils how to calculate certain addition exercises by means of Cuisenaire rods. Then she gave some addition exercises to her class as home work assignment. One pupil solved all the exercises using his own (mathematically correct) strategy and got the correct answers for all the exercises. However, the teacher marked all his answers as wrong. When the child's mother came to argue with the teacher about her judgment, the teacher's respond was that the use of the Cuisenaire rods is essential part of the final result. Here, the teacher did not distinguish between the pedagogical content and the mathematical content. I consider the failure to understand the conceptual difference between the two as a special case of pseudo conceptual understanding.

(III) The third anecdote is taken from a study which investigated the *ability of prospective teachers to prove or refute arithmetic statements* (Tirosh, 2002). Proof of mathematical statements is one of the issues discussed quite often in mathematical teacher training. We strongly emphasize that checking some examples is not enough to establish the validity of a *universal statement*. The students realize that a proof should be general but their impression is that a typical feature of being general is *the use of letters*. In the above study the researcher asked the students about refuting a universal statement. The question was whether, in order to refute a universal statement it is enough to point at one example for which the statement is not true. A significant number of students claimed that checking one example is not enough to refute a universal statement. Many students claimed that letters must be used in order to establish a proof. I suggest that this result indicates lack of understanding of the essence of proof. The students were trying to identify proofs by their superficial form. They failed to rely on meaning. Hence, I classify this result as pseudo conceptual behavior.

The pseudo conceptual behavior is directly related to one of the workshops in the above mentioned Symposium on Elementary Mathematics Teaching, SEMT 11 (August 21–26, 2011): Contribution of the theory of didactical situations to mathematics education (Chopin&Novotná, 2011). It is claimed there that during the student — teacher discourse, sometimes, "the teacher begs for a sign that the student is following him" (p. 362). Eventually, the teacher is lowering the intellectual demands which were originally involved in the question. At a certain stage, the student can give an answer to the teacher's question relying only on superficial indicators without having any knowledge of the subject involved. This is called by Brousseau & Sarrazy (2002) "the Topaze effect".

A student's meaningless behavior is demonstrated in a magnificent way in the classical play by Moliere Le Bourgeois Gentilhomme (1670). I first dealt with it in Vinner (1997). I would like to present it here again because it is directly related to Chopin and Novotná (2011) as well as to students who learn certain topics in a meaningless way and then teach them. I claimed earlier that it happens with elementary mathematics teachers and that it is disastrous. In the first excerpt from Le Bourgeois Gentilhomme, the student (Mr. Jourdain) is talking to his teacher (the philosopher).

Mr. Jourdain: *I am in love with a lady of quality and I want you to help me to write her a little note I can let fall at her feet.* Philosopher: *Very well.* Mr. Jourdain: *That's the correct thing to do, isn't it?* Philosopher: *Certainly. You want it in verse no doubt?* Mr. Jourdain: *No. No. None of your verse for me.* Philosopher: *You want it in prose then?* Mr. Jourdain: *No. I don't want it in either.* Philosopher: *But it must be one or the other.* Mr. Jourdain: *Why?* Philosopher: *Because, my dear sir, if you want to express yourself at all there's only verse or prose for it.* Mr. Jourdain: *Only prose or verse for it?* Philosopher: *That's all, sir. Whatever isn't prose is verse and anything that isn't verse is prose.* Mr. Jourdain: *And talking, as I am now, which is that?* Philosopher: *That is prose.* Mr. Jourdain: *You mean to say that when I say "Nicole, fetch me my slippers" or "give me my night -cap" that's prose?* Philosopher: *Certainly, sir.* Mr. Jourdain: *Well, my goodness! Here I have been talking prose for forty years and I have never known it.*

Till a certain point, the above conversation sounds like meaningful communication. ("You want it in verse no doubt? — No. No. None of your verse for me.") Only at a crucial point it turns out that the student has no idea of the topic discussed. ("No. I don't want it in either.") There is also an implicit criticism about teaching in the above dialogue and this is quite relevant to the Topaze effect. Because of the student's ignorance, the teacher is forced to teach in an inadequate way. The distinction between verse and prose is a distinction between literary forms. It is quite ridiculous to apply it to everyday language. The philosopher did it as a kind of didactic simplification. The student, lacking the required background to assimilate this (or in the terminology used in this paper, it is not within his ZPD) will handle it in a pseudo-conceptual mode. This mode will prevail also when the student becomes a teacher. Thus, in the second excerpt Mr. Jourdain becomes a teacher and he tries to teach what he just learned to his wife.

Mr. Jourdain: …*Do you know what you are doing — what you are talking at this very moment?* Mrs. Jourdain: *I'm talking plain common sense — you ought to be mending your ways.* Mr. Jourdain: *That's not what I mean. What I am asking is what sort of speech are you using?* Mrs. Jourdain: *Speech. I'm not making a speech. But what I'm saying makes sense and that's more than can be said for your goings on.* Mr. Jourdain: *I'm not talking about that. I'm asking what I am talking now. The words I am using — what are they?* Mrs. Jourdain: *Stuff and nonsense!* Mr. Jourdain: *Not at all. The words we are both using. What are they?* Mrs. Jourdain: *What on earth are they?* Mr. Jourdain: *What are they called?* Mrs. Jourdain: *Call them what you like.* Mr. Jourdain: *They are prose, you ignorant creature!*

It seems to me that no additional comments are needed here since the text speaks for itself.

9 Epilogue

I would like to conclude my paper with two comments. The first one is related to the elementary teachers. Observing them in their classes indicates that in most cases they are dedicated people. They do their best to teach mathematics. Sometimes, their best is not good enough mathematically. However, it is useless and pointless to request more than their best. The second comment relates to us, the mathematics education community. My recommendation to focus on aspects of the elementary teacher performance which are not content knowledge or pedagogical content knowledge may look to some of us as a *threat*. Improving mathematical achievements is considered by many of us as our ultimate goal. Do I recommend considering other educational values as the ultimate goal of education? As a matter of fact I do, but it does not matter. There is nothing to worry about. The different educational systems (local, national and international) will not give up mathematical achievements as *a draconian filter* for further studying. Hence, improving mathematical achievements will still get the financial support that many of us look for, and mathematics education research will continue to focus on mathematical achievements. My recommendation, therefore, is only to look at things differently and, from time to time, to remind ourselves the real goal of education — an educated adult.

Appendix I: A questionnaire the purpose of which is to find out how elementary mathematics teacher view mathematics

As I explained in section 5, I made the questionnaire while being guided by two different intentions. The first one was to call or invite a definition which will reflect somehow the dictionary definition. Namely, Mathematics is the science of something. The other intention was to point at a different answer to the question what mathematics is, an answer with a seemingly negative connotation and to see how the respondents cope with it. So here is the questionnaire:

In a conversation between a history teacher and a mathematics teacher at the teacher room the history teacher said: When I studied in high school I was interested in the natural sciences. I studied physics and biology. I understood that physics describes the laws of the physical world (movement, electricity, light, heat and so on) and biology describes the world of animals and plants. On the other hand, my impression about mathematics was that it is a collection of procedures, rules and formulas to be used in order to solve some typical questions given in homework assignments and in some crucial exams (midterm exams, final course exams and the matriculation exams).

What is your reaction to the history teacher's claim?

1. I definitely agree. 2. I agree with a certain reservation. 3. I do not agree but I understand how such an impression was created. 4. I absolutely disagree.

Please, explain your answer! In case you do not agree with the history teacher's claim — please, try to say what mathematics is in your opinion.

Because of space restriction I am not presenting here the full distribution and the analysis of the answers.

Appendix II: What is mathematics?

The answer to this question depends on the eyes of the beholder. Everybody can express his or her views about it. Thus, perhaps we should ask who is authorized to answer it and to give an authorized answer. In my opinion, the authorized person is a research mathematician. I am aware of the fact that different research mathematicians might give different answers to the question. Nevertheless, since I promised earlier to suggest a definition I am presenting it here, as a mathematical research survivor whose domain was mathematical logic and theory of models. I assume that my mathematical logic background has a certain impact on my perception of mathematics.

If we look at courses which mathematics majors are supposed to complete for their Bachelor degree or Master degree we find out that many of these courses include the word "theory" in their title. For instance: Group Theory, Ring Theory, Number Theory, Game Theory and so on and so forth. Each of these theories relates to a specific *mathematical structure*. So, what is a mathematical structure? Each mathematical structure includes a set of elements about which various statements are made. It can be the set of numbers in the case of Number theory, it can be a set of a group elements in case of group theory, a set of vectors in case of Vector Analysis or a set of complex functions in case of The theory of Functions of Complex Variable and so on and so forth. The elements of the set of the mathematical structure are regarded as *mathematical objects*. Some mathematicians consider them as *abstract objects*. Since I do not want to elaborate on this notion (this may lead me to a philosophy of mathematics discussion) I will stick to the previous notion — mathematical objects. In order to ease my presentation, let us observe Peano's Arithmetic as a *generic example*. The set of natural numbers is the set of mathematical elements of this structure. In this set of elements there are two distinguished elements, 0 and 1 (this version of Peano's Arithmetic includes 0. There is another version that starts the sequence of natural numbers with 1). In addition to the set of numbers the structure has also some operations: unary operation (the successor), binary operations (addition and multiplication) and several more which can be defined by these operations. It has also relations: The equality relation (=, a binary relation), the "less than" relation (<, a binary relation) and several more that can be defined by means of these; for instance, the divisibility relation ($a \mid b$, a divides b, a binary relation), the "being a prime" relation (a unary relation) and so on and so forth. The Peano's Arithmetic is, in fact, the collection of all true statements about the above structure. For instance, it includes the statement that there are infinitely many prime numbers. It also includes the statement that any (natural) number can be expressed as a sum of 4 squares of (natural) numbers, and so on. The crucial question here is how the collection of all true arithmetical statements can be reached. The answer is well known to any mathematics major — by the deductive method. This means that we start with some statements about which we know they are true (we try to minimize their number) which are called *axioms*. Then by certain rules of inference we derive additional statements. The inference rules guarantee that if we start from true statements we obtain additional true statements.

I suggest the above as a characterization of specific examples presented to any mathematics major all over the world. I am aware of the fact that it is an over simplification. Because of that it is also inaccurate. However, in order to please the rigorous mathematician I will have to expand it in such a way that it will include a chapter from mathematical logic about first order predicate calculus and also a chapter from model theory and more. So, a reader with a sufficient background will be able to accept it as a schematic illustration. For readers who lack the sufficient background, I am afraid, it will not make sense. Thus, many readers who lack the sufficient background will form their idea about what mathematic is by relying on their own experience. This brings us back to section 5 and the idea that elementary teachers may have about what mathematics is. When discussing this question with people who did not graduate in mathematics (as the majority of elementary mathematics teachers) one can hear that they consider Excel and Sudoku as part of mathematics, they consider Cuisenaire Rods as part of mathematics, they consider other concrete materials as part of mathematics, as well as mathematical education software. They do not distinguish between means to study mathematics and mathematics. Thus also various types of word problems and mathematical riddles are considered as mathematics, where no distinction is made between the mathematics which is required in order to solve these problems and the problems themselves.

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